This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

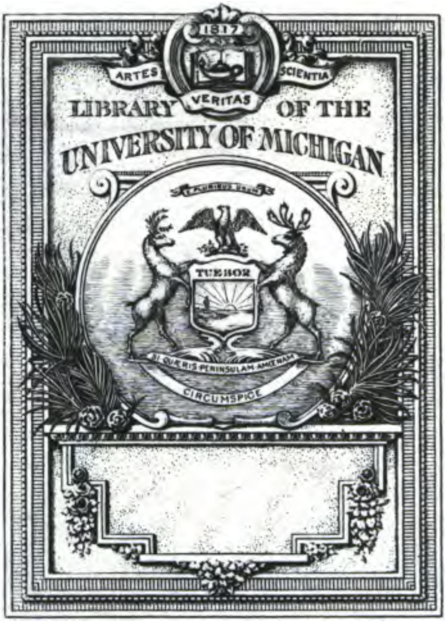
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`


1817
ARTES
SCIENTIA
VERITAS
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
E PLURIBUS UNUM
TUEBOR
SI QUAERIS PENINSULAM AMOENAM
CIRCUMSPICE

QA
531
.G82

ELEMENTS

OF

PLANE AND SPHERICAL TRIGONOMETRY:

WITH

THEIR APPLICATIONS

TO

HEIGHTS AND DISTANCES, PROJECTIONS OF THE SPHERE, DIALLING, ASTRONOMY, THE SOLUTION OF EQUATIONS, AND GEODESIC OPERATIONS.

Gilbert

BY

OLINTHUS GREGORY, LL. D.

Of the Royal Military Academy,

VICE-PRESIDENT OF THE LONDON PHILOSOPHICAL SOCIETY, AND HONORARY MEMBER OF THE LITERARY AND PHILOSOPHICAL, AND OF THE ANTIQUARIAN, SOCIETIES AT NEWCASTLE UPON TYNE.

LONDON:

PRINTED FOR BALDWIN, CRADOCK, AND JOY,
47, PATERNOSTER-ROW;
W. BLACKWOOD, EDINBURGH; AND J. CUMMING,
DUBLIN.

1816.

Printed by C. Baldwin,
New Bridge-street, London.

Hist. of Sci.
Dudley
5-6-46
54373

PREFACE.

HE who wishes to acquire any other reputation than that of being useful, will not devote his time to the preparation of an elementary work. Such, however, is the humble honour after which I here aspire: and if the small volume I now lay before the public shall contribute, in any degree, to increase the knowledge and improve the taste of the mathematical student in the earlier stages of his progress, I shall have attained my object.

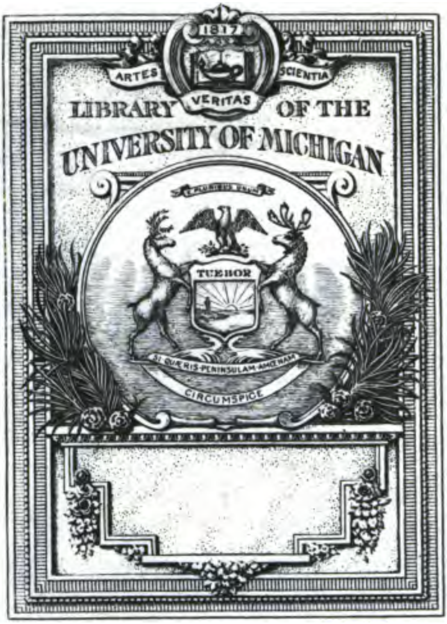
Of late years the authors of elementary treatises on different branches of mathematics, and especially Dr. *Hutton* and Mr. *Bonnycastle*, in their compendious manuals for popular use, have shown that it is possible, by judicious arrangement and selection, to compress much interesting, valuable, and scientific matter, within the compass of a small duodecimo volume. There can, I think, be no doubt that to the extensive circulation of this class of books, together with the stimulus furnished by the annual problems in the *Ladies'* and *Gentleman's Diaries*, and those proposed in the Senate House at Cambridge, must principally be ascribed the circumstance that mathematical knowledge, to a certain extent, is more widely diffused over the middle classes of society in this country, than in any other part of Europe.

What, however, has been so successfully effected in other departments of mathematics has not yet been attempted with regard to Trigonometry. We have some excellent works on this subject, whose value it would ill become me to depreciate. But such of them as go extensively into the business of Trigonometry and its applications are too large and expensive for general circulation; while others, being confined almost entirely to the elements, exclusive of the applications, must of necessity be restricted, both in point of circulation and utility. There is one treatise, that of *Emerson*, which is a most copious store-house of curious and elegant theorems: but they are so obscured by a defective notation, that the perusal of greater part of the book must, to a mathematical student, be as perplexing as the solution of a perpetual string of enigmas.

It has been my aim to steer into a middle course, between that in which is presented a mere commonplace book of principles and theorems, and that which, by leading far into the detail of multifarious processes and methods, precludes the study of the science, except by the sacrifice of much time and expense. In order to this I have endeavoured to be select in my materials, and have, for the most part, observed unity of method. A book of three times the size and price might have been drawn up with far less intellectual labour (for the fatigue of *selecting* from a fund of valuable materials, and casting the result into one mould, is not slight); but such a work would not have tended to accomplish the purpose I have in view. By adopting a small type and a full page, and confining myself chiefly to the analytical mode of investigation, I have been able to introduce, and I shall rejoice if it be thought I have treated perspicuously, a greater variety of the applications of plane and spherical Trigonometry than are to be found in any other work on the subject with which I am acquainted. In the first three chapters I have exhibited the theory of

Plane Trigonometry *geometrically*, and have shown the application of that theory to the logarithmic solutions of the usual cases into which this portion of the subject is conveniently distributed. I have endeavoured to conduct these introductory inquiries with the utmost perspicuity; that the student, by obtaining a thorough comprehension of the principal topics of research, and by seeing a little of their utility, may enter with the greater relish upon the subsequent investigations; and by tracing the correspondence of these results with such as will afterwards appear in the *analytical* theory of Plane Trigonometry, may be prepared to lean with full confidence upon the analytical formulæ that are in other places to be laid before him. The deductions from theory in the succeeding chapters are usually obtained by analytical processes; and their utility is shown in the logarithmic and trigonometric solution of a great number of problems, classified under the particular heads of the several applications, as specified in the table of contents. The whole, except what relates to the minute variations of the sides and angles of triangles, and the differential analogies which apply to them, may, I am persuaded, be readily comprehended by any person who is tolerably conversant with the elements of Geometry and Algebra.

In the composition of the work I have freely availed myself of all such matter as was likely to answer my purpose, especially in the productions of foreign mathematicians. The plan and method are of course my own: the materials have been collected, almost of necessity, from all quarters. In addition to the acknowledgments which will occur in different parts of this little volume, it would be unjust not to say here, that the theory of Projections, the general problem in reference to Dialling, and the comprehensive table of Differential Equations for the variations of triangles, are taken, simply with such alterations as fitted them better for general



1817
ARTES VERITAS SCIENTIA
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
E PLURIBUS UNUM
TUEBOR
SI QUÆRIS PENINSULAM AMŒNAM
CIRCUMSPICE

QA
531
.G82

this, as in every other region of abstract science, so in trigonometry, must, if the pursuit is intended to be at all extensive, be conducted, for the most part, (I do not say, exclusively) according to the principles of analysis.

OLINTHUS GREGORY.

Royal Military Academy,
Woolwich, Feb. 1816.

CONTENTS.

Page

Chap. I. Preliminary definitions and principles . . 1

II. General properties and mutual relations of the lines and angles of circles and plane triangles 9

III. Solutions of the several cases of plane triangles 24

IV. Plane trigonometry considered analytically 36

V. Application of plane trigonometry to the determination of heights and distances . 61

VI. Spherical trigonometry, in six sections

Sect. 1. Fundamental principles, general properties, and formulæ 79

Sect. 2. Right angled spherical triangles . 86

Sect. 3. Oblique angled ditto 89

Sect. 4. Analogies of Napier 94

Sect. 5. Mnemonics of spherical trigonometry 98

Sect. 6. Areas of spherical triangles and polygons, and measure of solid angles 101

VII. Logarithmic computation of spherical triangles 111

VIII. Projections of the sphere, in three sections

Sect. 1. Astronomical definitions . . . 117

Sect. 2. Orthographic projection . . . 122

Sect. 3. Stereographic projection . . . 125

IX. Principles of dialling 134

X. Astronomical problems 151

XI. Minute variations of triangles, with a table of differential equations 187

XII. Miscellaneous problems, in two sections

Sect. 1. Problems with solutions 206

Sect. 2. Problems without solutions . . . 232

ERRATA.

Page 8, line 4, *for* $\genfrac{}{}{0pt}{}{\text{rad}^2}{\text{sin}}$ *read* $\frac{\text{rad}^2}{\text{sin}}$.

14. Let a line be drawn from C to the point of contact A, in the first figure.

29, line 8, *for* answer *read* answers

37, line 2, *for* and is *read* it is

41, line 3, *for* $\genfrac{}{}{0pt}{}{a}{c}$ *read* $\frac{a}{c}$.

156, *Ex*. 1. Add, In the *Connaissances des Tems*, pour l'an 1817, there is given a table by M. Burckhardt to shorten the computation of this problem.

167, line 9 from bottom, *for* limited, *read* limits

212, last line but one, *for* $+ 2\sqrt{\tfrac{1}{2}p}$, *read* $\mp 2\sqrt{\tfrac{1}{2}p}$.

TRIGONOMETRY.

CHAPTER I.

Preliminary Definitions and Principles.

1. THE word *Trigonometry* signifies the measure of triangles. But in an enlarged sense we comprehend under this name, the science by which we are enabled to determine the positions and dimensions of the different parts of space, by means of the previous knowledge of some of those parts.

2. If we conceive any different points whatever, posited in space, to be joined one to another by right lines, there will be presented to our consideration three things: 1st. The *lengths* of those lines. 2dly. The *angles* they respectively form. 3dly. The angles formed respectively, by *planes* in which those lines are, or may be imagined to be, comprehended. On the comparison of these three objects depends the solution of the various questions which can be proposed, as to the measure of extension, and of its parts.

3. The intersections of three or more lines in one and the same plane, constitute angles limited by right lines, and plane or rectilinear triangles, or polygons susceptible of being resolved into triangles. And the intersections of three or more planes, form plane angles and triedral or polyedral surfaces, the determination of the magnitudes and relations of which is facilitated by re-

ference to the surface of a sphere. Hence mathematicians have been led to attempt the solution of two general problems.

I. Knowing three of the six things, whether angles or sides, which enter the constitution of a rectilinear triangle, to determine the other three; when it is possible.

II. Knowing three of the six things which compose a triangle formed on the surface of a sphere, by the intersections of three planes, which also meet in the centre of that sphere, to determine the other three, when possible.

The resolution of the first of these general problems appertains to *plane* or *rectilinear trigonometry:* that of the second, to *spherical trigonometry.*

4. Lines and angles, being magnitudes of different kinds, do not admit of comparison. It becomes necessary, therefore, to have recourse to quantities of an intermediate kind, akin to the one, yet having an obvious dependence upon the other, and serving as a common vinculum. Such are the *lineo-angular* quantities denominated *sines, tangents,* &c. which we are about to define. They are *lines,* but lines which admit of being measured only by parts of an assigned line, *the radius of a certain circle;* and lines which at the same time depend altogether, for their value, upon arcs of that circle, which arcs are, themselves, adequate measures of the angles included between the radii which limit such arcs.

5. By means of this happy invention of intermediate quantities, the business of trigonometry is greatly facilitated. For, by imagining a perpendicular let fall from the vertical angle of an oblique angled plane triangle upon the base, or base prolonged, it will at once be manifest that the resolution of triangles generally, may be referred to that of right angled triangles. Thus, assuming for a term of comparison, the hypothenuse of a right angled triangle, equal to unity, for example, computing the bases and perpendiculars of all possible right

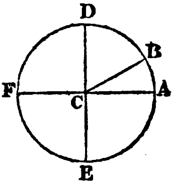
angled triangles having the assigned hypothenuse, and arranging them in different columns of a table, the magnitudes of the different parts of any proposed triangle, would become determinable upon the known principles of similar triangles. Such a table as this, would, as will soon be seen, be no other than a table of natural sines.

PLANE TRIGONOMETRY.

6. *Plane trigonometry* is that branch of mathematics, by which we learn how to determine or compute three of the six parts of a plane, or rectilinear triangle, from the other three; when that is possible.

This limitation is necessary, although there is only one case in which it can occur, namely, that in which the three *angles* of a rectilinear triangle are given. For, it is plain from Euc. vi. 4, that while the three angles of a triangle remain the same, the sides, though retaining the same mutual relation, may be greater or less, in all conceivable proportions.

7. *Lemma* I. Let ACB be a rectilinear angle: if, about the point C as a centre, and with any distance, or radius, CA, a circle be described, intersecting CA, CB, the right lines that include the angle ACB, in A and B; the angle ACB will be to four right angles, as the arc AB to the whole circumference of the circle ADFE.



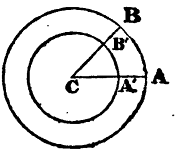
Produce AC to meet the circle in F, and through the centre C draw DE, another diameter, to meet the circle in D, E.

Then, Euc. vi. 33, ang. ACB : right ang. ACD :: arc AB : arc AD,

and, Euc. v. 4, cor. quadrupling the consequents, we have, angle ACB : 4 right angles :: arc AB : 4AD, that is, to the whole circumference.

8. *Lemma* II. Let ACB be a rectilinear angle: if, about C as a centre with any two distances CA′, CA, two circles be described, meeting CA, CB, in A′, B′, A, B; the arc AB will be to the whole circumference of which it is a part, as the arc A′B′ to the whole circumference of which it is a part.

By, lem. 1, arc AB : whole circum.
:: angle ACB : 4 right angles,
and, arc A′B′ : its whole circum.
:: angle A′CB′ : 4 right angles.
Therefore, arc AB : whole circum.
:: arc A′B′ : whole circum.
of its respective circle.



Definitions.

9. Let ACB be a rectilinear angle, if about C as a centre, with any radius CA, a circle be described, intersecting CA, CB, in A, B, the arc AB is called the *measure* of the angle ACB.

10. The circumference of a circle is supposed to be divided, or to be divisible, into 360 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* each of these into 60 equal parts, called *seconds;* and so on, to the minutest possible subdivisions. Of these, the first is indicated by a small circle, the second by a single accent, the third by a double accent, &c. Thus, $47^\circ\, 18'\, 34''\, 45'''$, denotes 47 degrees, 18 minutes, 34 seconds, and 45 thirds. So many degrees, minutes, seconds, &c. as are contained in any arc, of so many degrees, minutes, seconds, &c. is the angle of which that arc is the measure said to be. Thus, since a quadrant, or quarter of a circle, contains 90 degrees, and a quadrantal arc is the measure of a right angle, a right angle is said to be one of 90 degrees.

11. The *complement of an arc* is its difference from a quadrant; and the *complement of an angle* is its difference from a right angle.

12. The *supplement of an arc* is its difference from a semicircle; and the *supplement of an angle* is its difference from two right angles.

13. The *sine* of an arc is a perpendicular let fall from one extremity upon a diameter passing through the other.

14. The *versed sine* of an arc is that part of the diameter which is intercepted between the foot of the sine and the arc.

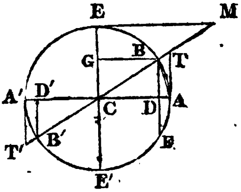
15. The *tangent* of an arc is a right line which touches it in one extremity, and is limited by a right line drawn from the centre of the circle through the other extremity.

16. The *secant* of an arc is the sloping line which thus limits the tangent.

17. These are also, by way of accommodation, said to be the sine, tangent, &c. of the angle measured by the aforesaid arc, to its determinate radius.

18. The *cosine* of an arc or angle, is the sine of the complement of that arc or angle: the *cotangent* of an arc or angle is the tangent of the complement of that arc or angle. The *co-versed sine*, and *co-secant* are defined similarly.

To exemplify these definitions by the annexed diagram: let AB be an assumed arc of a circle described with the radius AC, and let AE be a quadrantal arc; let BD be demitted perpendicularly from the extremity B upon the diameter AA′; parallel to it let AT be drawn and limited by CT: let GB and EM be drawn parallel to AA′, the latter being limited by CT or CT produced. Then BE is the *complement* of BA, and angle BCE the *complement* of angle BCA; BEA′ is the *supplement* of BA, and angle BCA′ the *supplement* of BCA; BD is the *sine*, DA the *versed sine*, AT the *tangent*, CT the *secant*, GB the



cosine, GE the *coversed sine,* EM the *cotangent,* and CM the *cosecant,* of the arc AB, or, by convention, of the angle ACB.

Note. These terms are indicated by obvious contractions:

Thus, for sine of the arc AB we use sin AB,
tangent .... ditto ...... tan AB,
secant .... ditto ...... sec AB,
versed sine ditto ...... versin AB,
cosine...... ditto ...... cos AB,
cotangent .. ditto ...... cot AB,
cosecant.... ditto ...... cosec AB,
coversed sine ditto ...... coversin AB.

Corollaries from the Definitions.

19. (A). Of any arc, less than a quadrant, the arc is less than its corresponding tangent; and of any arc whatever, the chord is less than the arc, and the sine less than the chord.

For, in the preceding diagram, the circular sector CAB is less than the triangle CAT, the former being contained within the latter. That is, by the rules for mensuration of surfaces, $\frac{1}{2}$CA × arc AB is less than $\frac{1}{2}$CA × tan AT; whence, dividing by $\frac{1}{2}$CA, there results arc AB less than tangent AT.

In a similar way it may be seen that chord AB, is less than arc AB, and sine BD, less than chord AB. The same is also evident from the consideration that a right line AB is less than any curve line terminated by the same points, and the perpendicular BD less than the hypothenuse AB, of a right angled triangle ADB.

(B). The sine BD of an arc AB, is half the chord BF of the double arc BAF.

(C). An arc and its supplement have the same sine, tangent, and secant. (The two latter, however, are affected by different signs, + or −, according as they appertain to arcs less or greater than a quadrant: the reasons of this will be explained in a subsequent chapter.)

(D). When the arc is evanescent, the sine, tangent, and versed sine, are evanescent also, and the secant becomes equal to the radius, being its minimum limit. As the arc increases from this state, the sines, tangents, secants, and versed sines, increase; thus they continue, till the arc becomes equal to a quadrant AE, and then the sine is in its maximum state, being equal to radius, thence called the *sine total;* the versed sine is also then equal to the radius; and the secant and tangent becoming incapable of mutually limiting each other, are regarded as infinite.

(E). The versed sine of an arc, together with its cosine are equal to the radius. Thus, AD + BG = AD + DC = AC. (This is not restricted to arcs less than a quadrant, as will be seen in the chapter on analytical plane trigonometry.)

(F). The radius, tangent, and secant, constitute a right angled triangle CAT. The cosine, sine, and radius, constitute another right angled triangle CDB, similar to the former. So again, the cotangent, radius, and cosecant, constitute a third right angled triangle MEC, similar to both the preceding. Hence, when the sine and radius are known, the cosine is determined by Euc. i. 47. The same may be said of the determination of the secant, from the tangent and radius, &c. &c. &c.

(G). Further, since CD : DB :: CA : AT, we see that the tangent is a fourth proportional to the cosine, sine, and radius.

Also, CD : CB :: CA : CT; that is, the secant is a third proportional to the cosine and radius.

Again, CG : GB :: CE : EM; that is, the cotangent is a fourth proportional to the sine, cosine, and radius.

And, BD : BC :: CE : CM; that is, the cosecant is a third proportional to the sine and radius.

(H). Thus, employing the usual abbreviations, we should have

1. $\cos = \sqrt{(\text{rad}^2 - \sin^2)}$. 2. $\tan = \sqrt{(\sec^2 - \text{rad}^2)}$.
3. $\sec = \sqrt{(\text{rad}^2 + \tan^2)}$. 4. $\text{cosec} = \sqrt{(\text{rad}^2 + \cot^2)}$.
5. $\tan = \frac{\text{rad} \times \sin}{\cos}$. 6. $\cot = \frac{\text{rad} \times \cos}{\sin}$.
7. $\sec = \frac{\text{rad}^2}{\cos}$. 8. $\text{cosec} = \frac{\text{rad}^2}{\sin}$.

These, when unity is regarded as the radius of the circle, become

1. $\cos = \sqrt{(1 - \sin^2)}$. 2. $\tan = \sqrt{(\sec^2 - 1)}$.
3. $\sec = \sqrt{(1 + \tan^2)}$. 4. $\text{cosec} = \sqrt{(1 + \cot^2)}$.
5. $\tan = \frac{\sin}{\cos}$. 6. $\cot = \frac{\cos}{\sin}$. 7. $\sec = \frac{1}{\cos}$. 8. $\text{cosec} = \frac{1}{\sin}$.

20. From these and other properties and theorems, some of which will be demonstrated as we proceed, mathematicians have computed the lengths of the sines, tangents, secants, and versed sines, to an assumed radius, that correspond to arcs from 1 second of a degree, through all the gradations of magnitude, up to a quadrant, or 90°. The results of the computations are arranged in tables called *Trigonometrical Tables* for use. The arrangement is generally appropriated to two distinct kinds of these artificial numbers, classed in their regular order upon pages that face each other. On the left hand pages are placed the sines, tangents, secants, &c. adapted at least to every degree, and *minute*, in the quadrant, computed to the radius 1, and expressed decimally. On the right hand pages are placed in succession the corresponding *logarithms* of the numbers that denote the several sines, tangents, &c. on the respective opposite pages. Only, that the necessity of using negative indices in the logarithms may be precluded, they are supposed to be the logarithms of sines, tangents, secants, &c. computed to the radius 10000000000. The numbers thus computed and placed on the successive right hand pages are called *logarithmic* sines, tangents, &c. The numbers of which these are the logarithms, and which are arranged on the left hand

pages, are called *natural* sines, tangents, &c. In the small pocket tables these are usually omitted, and the logarithmic alone retained, as the most useful and expeditious in operation.

Of the various trigonometrical tables which have been published at different times, those which deserve the warmest recommendation, as most accurate and best fitted for general use are "*The Mathematical Tables" of Dr. Hutton*, in one vol. royal 8vo.; and the stereotyped "*Tables Portatives de Logarithmes, par Francois Callet*," printed also in royal 8vo. in 1795. Dr. Hutton's work contains a copious and valuable introduction, comprizing the history, nature, construction and use of logarithmic and trigonometrical tables. The introduction to Callet's tables, likewise exhibits directions for their use, and some of the best formulæ employed in their construction.* Of small tables for the pocket the best with which I am acquainted are those of *Mr. Whiting*, and the stereotyped tables of the Rev. *F. A. Barker*.

CHAPTER II.

General Properties, and Mutual Relations of the Lines and Angles of Circles and Plane Triangles.

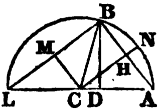
PROP. I.

1. THE chord of any arc is a mean proportional between the versed sine of that arc and the diameter of the circle.

---

* The nature and use of logarithms being fully explained in Dr. Hutton's valuable work, and, indeed, in every collection of logarithmic tables which a student ought to possess, I think it entirely unnecessary to occupy any portion of this introduction by an elucidation of the properties of those useful numbers.

In the marginal figure, AB is the chord, and AD is the versed sine of the arc AB. BL being joined the angle LBA in a semicircle is a right angle, and is therefore equal to angle BDA, BD being perpendicular to LA. Hence, the triangles BDA, LBA, are similar, and we have AD : AB :: AB : AL.



Prop. II.

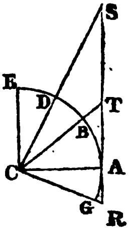
2. As radius : cosine of any arc :: twice the sine of that arc : the sine of double the arc.

In the preceding figure CH is the cosine of the arc BN, AB is twice the sine BH of that arc, and BD is the sine of AB the double arc. From the similar triangles ACH, ABD, we have AC : CH :: AB : BD.

Prop. III.

3. The secant of any arc is equal to the sum of its tangent, and the tangent of half its complement.

In the annexed diagram, where AB is the proposed arc, let the tangent TA be produced downwards till TA + AR = CT the secant. Then, since angle R is the complement of ACR, and of $\frac{1}{2}$ATC, it follows that ACR = $\frac{1}{2}$ATC, that is = $\frac{1}{2}$ comp. ACT. But AR is the tangent of ACR, or of the arc AG; whence the proposition is manifest.



Prop. IV.

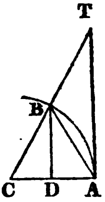
4. The sum of the tangent and secant of any arc, is equal to the tangent of an arc exceeding that by half its complement.

Produce AT, the tangent of the assumed arc, till the prolongation TS becomes equal to CT the secant, and join CS, intersecting the quadrantal arc AE in D. Then, because TS = TC, angle TCS = TSC = TCE, by reason of the parallels TS, CE. Hence $BD = DE = \frac{1}{2}$ comp. AB, and $AS = \tan AB + \sec AB = \tan AD = \tan (AB + \frac{1}{2}$ comp $AB)$.

Prop. V.

5. The chord of 60° is equal to the radius of the circle; the versed sine, and cosine of 60° are each equal to half the radius; and the secant of 60° is equal to double the radius.

Let AB be an arc of 60°, AB its chord, CD its cosine, AD its versed sine, CT its secant. Then,



1. Since $ACB = 60°$, and $CB = CA$, $A = B = \frac{1}{2}(180° - 60°) = 60°$. That is, the three angles of the triangle ABC are equal, and therefore the triangle is equilateral. Conseq. AB = radius.

2. $CD = DA = \frac{1}{2}CA$, because in an isosceles or equilateral triangle the perp. bisects the base.

3. $BAT =$ comp. $60° = BTA$; therefore $BT = BA$; and $TB + BC = TC = 2BC = 2$ radius.

Cor. If with the same radius an arc were described from centre B, then CD would become the sine of 30°, which is consequently half the radius.

Prop. VI.

6. *The tangent of an arc of 45° is equal to the radius.*

Suppose AB in the last figure but one were an arc of 45°; then would ACB be half a right angle, and consequently its complement ATC. The sides AT, and AC, opposite to those angles, would then be equal; that is, tan 45° = radius.

Cor. From this and prop. 5, it is evident that the sine of 30°, tangent of 45°, and secant of 60°, are in the ratio of the numbers 1, 2, and 4; or that tan 45° is a mean proportional between sin 30° and sec 60°.

PROP. VII.

7. The square of the sine of half any arc or angle is equal to a rectangle under half the radius and the versed sine of the whole; and the square of its cosine, equal to a rectangle under half the radius and the versed sine of the supplement of the whole arc or angle.

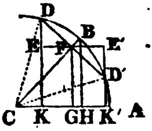
This will be at once obvious from the definitions, and the diagram to prop. 1, of this chapter. For $4AH^2 = AB^2 = AL \cdot AD = 2AC \cdot AD$. Whence $AH^2 = \frac{1}{2}AC \cdot AD$.

Also, $4CH^2 = LB^2 = AL \cdot LD = 2AC \cdot LD$. Whence $CH^2 = \frac{1}{2}AC \cdot LD$.

PROP. VIII.

8. The rectangle under the radius and the sine of the sum or of the difference of two arcs, is equal to the sum or the difference of the rectangles under their alternate sines and cosines.

Let AB and BD be two unequal arcs, of the circle whose radius is AC; and let $BD' = BD$. Then is AD the sum, and AD′ the difference, of the arcs AB and BD. Also, $BH =$ sine, $GH =$ cosine, of arc AB; $DF =$ sine, $CF =$ cosine, of arc BD; $DK =$ sine, $CK =$ cosine, of $AD = AB + BD$; $DK' =$ sine, $CK' =$ cosine, of $AD' = AB - BD$.



The triangles CBH, CFG, being similar, we have, $CB : BH :: CF : FG = EK$, whence $CB \cdot EK = BH \cdot CF$.

Again, the triangles CBH, DFE, are equiangular; for each has a right angle, H and E, and angle $CBH = DFE$, the former being equal to CFG, which is the complement to EFC, and the latter being also a complement to the same angle.

Hence, $CB : CH :: DF : DE$, and conseq. $CB \cdot DE = CH \cdot DF$.

But, since as above, $CB \cdot EK = BH \cdot CF$,

and $CB \cdot DE = CH \cdot DF$,

we have, by addition, $CB \cdot DK = BH \cdot CF + CH \cdot DF$, which is the first part of the proposition.

Again, since the triangles DFE, $D'FE'$, are equal as well as equiangular, $ED = E'D$: the preceding rectangles may, therefore, be expressed thus,

$$CB \cdot E'K' = BH \cdot CF,$$

$$\text{and } CB \cdot D'E' = CH \cdot DF,$$

of which the difference is $CB \cdot D'K' = BH \cdot CF - CH \cdot DF$, which is the second part of the proposition.

Cor. If the two arcs become equal, then we have for the sum, rad × sin $2AB$ = sin AB × 2 cos AB, agreeing with prop. 2 of this chapter.

Prop. IX.

9. The rectangle under the radius and the cosine of the sum or the difference of two arcs, is equal to the difference or the sum of the rectangles under their respective cosines and sines.

Recurring to the same diagram, we have from the similar triangles CBH, CFG,

$CB : CH :: CF : CG$; whence $CB \cdot CG = CH \cdot CF$:

and the similar triangles CBH, and DEF, give

$CB : BH :: DF : EF$ or KG; whence $CB \cdot KG = BH \cdot DF$.

The difference of these rectangles is,
$CB \cdot CK = CH \cdot CF - BH \cdot DF$; which is the first part of the proposition.

The equivalent rectangles to the preceding, are

$$CB \cdot CG = CH \cdot CF,$$

$$\text{and } CB \cdot K'G = BH \cdot DF;$$

the sum of which gives $CB \cdot CK' = CH \cdot CF + BH \cdot DF$, which is the second part of the proposition.

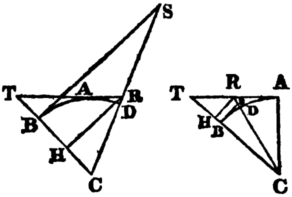
Cor. When AB and BC are equal, we have from this proposition rad × cos $2AB$ = cos ^2AB − sin ^2AB.

Remark. The preceding figure is adapted to the case where not only AB, and BD, but their sum AD is less than a quadrant. But the properties enunciated in these two propositions are equally true, let the magnitudes of the arcs, of their sum, and their difference, be what they may. This might be rendered evident by a suitable modification of the diagram; and will still farther appear in the fourth chapter of this work.

Prop. X.

10. As the difference or sum of the square of the radius and the rectangle under the tangents of two arcs, is to the square of the radius; so is the sum or difference of their tangents, to the tangent of the sum or difference of the arcs.

Let AB, AD, be the two arcs; AT, AR, their tangents; also let BS in the first figure be the tangent of the sum of those arcs, and BS in the second figure the tangent of their difference: and from the point R let RH, in both figures, be drawn parallel to BS, or perpendicular to the radius CB.



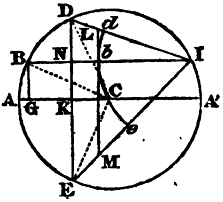
Then, because of the similar triangles TAC, THR, we shall have,

TC : CA :: TR : RH, whence TC . RH = CA . TR,
TC : TA :: TR : TH, whence TC . TH = TA . TR.

Each of the last equal rectangles being taken from the square of TC, there will remain, $TC^2 - TC \cdot TH$, or $TC \cdot (TC - TH)$, or $TC \cdot CH = TC^2 - TA \cdot TR$. Now, $TC \cdot CH$ (or $TC^2 - TA \cdot TR$) : $TC \cdot RH$ (or $CA \cdot TR$) :: CH : RH :: CB (or CA) : BS :: $CA^2 : CA \cdot BS$; whence, alternando, $TC^2 - TA \cdot TR : CA^2 :: CA \cdot TR : CA \cdot BS :: TR : BS$. Of this proportion, the first term $TC^2 - TA \cdot TR$, because $TC^2 = CA^2 + AT^2$, may also be expressed by $CA^2 + AT^2 - TA \cdot TR$, or by $CA^2 + AT^2 - TA\ (TA \pm AR)$, or lastly, by $CA^2 \mp AT \cdot AC$; whence the proposition is manifest.

PROP. XI.

11. **As the sum of the sines of two unequal arcs, is to their difference, so is the tangent of half the sum of those two arcs, to the tangent of half their difference.**



Let AE and AB be two unequal arcs, of which EK and BG, are the sines; and let EK be produced to cut the circle in D, and BI be drawn parallel to the diameter AA′. Draw ID, IE, from the centre I with the radius CA of the assumed circle, describe an arc *dbe*, and through *b* draw L*b*M parallel to DE. Then, it is evident that NE = KE + BG, is the sum of the sines of AE and AB; and ND = KE − BG = KD − KN, is their difference. Also, since BIE (at the circumference) = $\frac{1}{2}$BCE (at the centre); and BID = $\frac{1}{2}$BCD (Euc. iii. 20); *b*M is equal to the tangent of $\frac{1}{2}$(AE + AB), and *b*L = to the tangent of $\frac{1}{2}$(AE − AB), that is, of $\frac{1}{2}$(AD − AB.) But, by reason of the parallels DE, LM, we have EN : DN :: *b*M : *b*L; which is evidently the theorem enunciated above.

Cor. The sum of the cosines of two arcs, is to their difference, as the cotangent of half the sum of those two arcs, is to the [illegible]tangent of half their difference.

For, the cosines being the sines of the complements, it follows from the proposition that the sum of the cosines, is to their difference, as the tangent of half the sum of the complements, is to the tangent of half their difference. But half the sum of the complements of two arcs is the complement of half the sum of those two arcs, and half the difference of the complements is the same as ~~the complement of~~ half the difference; whence the truth of the corollary.

PROP. XII.

12. Of any three equidifferent arcs, it will be, as

radius, to the cosine of their common difference, so is the sine of the mean arc, to half the sum of the sines of the extremes; and, as radius, to the sine of the common difference, so is the cosine of the mean arc to half the difference of the sines of the two extremes.

Let AD′, AB, AD, (in the figure to prop. 8, of this chapter), be the three equidifferent arcs. Then DF = DF′, is the sine of their common difference, and CF its cosine. Also FG, being an arithmetical mean between the sines DK, D′K′, of the two extreme arcs, is equal to half their sum, and DE equal to half their difference. By reason of the similar triangles CBH, CFG, DFE,

we have, $CB : CF :: BH : FG$,
and $CB : DF :: CH : DE$;

which are the analogies in the proposition.

Cor. 1. From the preceding proportions, we have

$$FG = \tfrac{1}{2}(DK + D'K') = \frac{CF \,.\, BH}{CB}, \text{and } DE = \tfrac{1}{2}(DK - D'K') = \frac{DF \,.\, CH}{CB}.$$

$$\text{Therefore } DK + D'K' = \frac{2CF \,.\, BH}{CB}, \text{ and } DK - D'K' = \frac{2DF \,.\, CH}{CB}.$$

Cor. 2. Hence, if the mean arc AB be one of 60°, its cosine CH, will (prop. 5) be equal to $\frac{1}{2}$CB, and DK — D′K′ = DF: consequently DK will in that case equal DF + D′K′. From this conjointly with the preceding corollary result these two theorems:

(A). If the sine of the mean of three equidifferent arcs (radius being unity) be multiplied into twice the cosine of the common difference, and the sine of either extreme be deducted from the product, the remainder will be the sine of the other extreme.

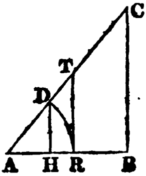
(B). The sine of any arc above 60°, is equal to the sine of another arc as much below 60°, together with the sine of its excess above 60°.

Remark. From this latter corollary, the sines below 60° being known, those of arcs above 60° are determinable by addition only.

Thus, $\sin 60^\circ\, 1' = \sin 59^\circ\, 59' + \sin 1'$,
$\sin 60^\circ\, 2' = \sin 59^\circ\, 58' + \sin 2'$,
$\sin 60^\circ\, 3' = \sin 59^\circ\, 57' + \sin 3'$,
$\sin 64^\circ\, 0' = \sin 56^\circ\, 0' + \sin 4^\circ$,
&c. &c. &c.

PROP. XIII.

13. In any right angled triangle, the hypothenuse is to one of the legs, as the radius to the sine of the angle opposite to that leg; and one of the legs is to the other, as the radius to the tangent of the angle opposite to the latter.



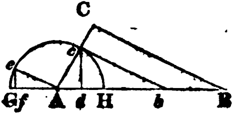
Let ABC be a triangle, right-angled at B, and let AR on the leg AB, be the radius of the tables. With centre A and radius AR, describe an arc to cut the hypothenuse in D, and draw DH, TR, perpendicular to AB. Then DH is the sine, TR the tangent, and AT the secant, of the arc DR, or angle A: and the similar triangles ABC, ART, AHD, give

AC : CB :: AD : DH :: rad : sin A;
and AB : BC :: AR : RT :: rad : tan A.

Cor. To the hypothenuse as a radius, each leg is the sine of its opposite angle; and to one of the legs as a radius, the other leg is the tangent of its opposite angle, and the hypothenuse is the secant of the same angle.

PROP. XIV.

14. In any plane triangle, as one of the sides, is to another, so is the sine of the angle opposite to the former, to the sine of the angle opposite to the latter.



Let ABC be a plane triangle, in which it is required to determine the relation which subsists between the sides AC and BC. With the angular point A as a centre, and the distance AC, equal to the radius of the tables, describe the semicircle G*ec*H. From *c* draw *cb* parallel to CB, and *cd* perpendicular to AB: draw also *e*A parallel to CB, and from the point of intersection *e* (of that line with the circle) demit the perpendicular *ef* on BA produced. Then *cd* is the sine of the angle CAB, to the radius A*c*, and *ef* is the sine of the angle *e*AG = angle CBA, to the same radius.

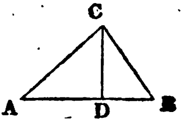
But AC : BC :: A*c* : *bc*, because of the parallels BC and *bc*,
:: A*e* : *bc*, because A*e* = A*c*,
:: *ef* : *cd*, because of the similar triangles A*ef*, *bcd*.

That is, AC : BC :: sin B : sin A.

In a similar manner it may be shewn that
AC : AB :: sin B : sin C, and AB : BC :: sin C : sin A.

And by drawing a figure for each case, it will be seen that the circumstance of any one of the angles being *obtuse* will make no difference in the demonstration.

Otherwise.



From C the vertical angle of the triangle let fall the perpendicular CD upon AB, or AB produced, according as the angles A and B are both acute, or one obtuse.

Then (prop. 13) AC : CD :: rad : sin A;
also CD : CB :: sin B : rad;
∴ ex æquo pertur. AC : CB :: sin B : sin A.

PROP. XV.

15. In any plane triangle it will be, as the sum of the sides about the vertical angle, is to their difference, so is the tangent of half the sum of the angles at the base, to the tangent of half their difference.

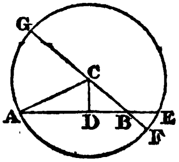
By the preceding prop. AC : BC :: sin B : sin A,
∴ by comp. and div. AC + BC : AC − BC :: sin B + sin A : sin B − sin A.

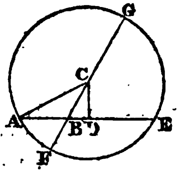
But, (prop. 11) $\sin B + \sin A :: \sin B - \sin A :: \tan \frac{1}{2}(B + A) : \tan \frac{1}{2}(B - A)$,

$\therefore AC + BC : AC - BC :: \tan \frac{1}{2}(B + A) : \tan \frac{1}{2}(B - A)$.

Prop. XVI.

16. In any plane triangle it will be, as the base, to the sum of the two other sides, so is the difference of those sides, to the difference of the segments of the base made by a perpendicular let fall from the vertical angle.

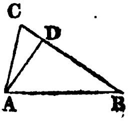




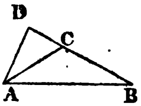
From the centre C (namely, the vertical angle of the triangle ABC) with the distance of the greater side AC describe the circumference of a circle, meeting AB, CB, produced, in the points E, F, G. Then, it is obvious that GB is equal to the sum of the sides, AC, CB, and FB equal to their difference. And because CD is perpendicular to AE, AD is equal to DE, (Euc. iii. 3). Wherefore, of the two AB, BE, one is the sum of the segments of the base DA, DB, and the other their difference. But, (Euc. iii. 35) the rectangle under GB, and BF, is equal to the rectangle under AB and BE. Consequently (Euc. vi. 16) $AB : GB :: BF : BE$; which agrees with the proposition.

Prop. XVII.

17. In any plane triangle it will be, as twice the rectangle under any two sides, is to the difference of the sum of the squares of those two sides and the square of the base, so is the radius to the cosine of the angle contained by the two sides.



Let ABC be a plane triangle. From A, one extremity of the base AB, draw AD perpendicular to the opposite side BC: then (Euc. ii. 12, 13) the difference of the sum of the squares of AC, CB, and the square of

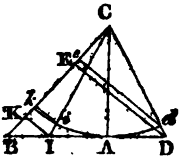


the base AB, is equal to twice the rectangle BC . CD. But twice the rectangle BC . CA, is to twice the rectangle BC . CD, that is, to the difference of the sum of the squares of AC, BC, and the square of AB, as CA to CD; that is, (prop. 13) as radius to the sine of CAD, or, as radius to the cosine of ACB.

Cor. When unity is assumed as radius, we have $\cos C = \frac{AC^2 + BC^2 - AB^2}{2CB . CA}$.

PROP. XVIII.

18. As the sum of the tangents of any two unequal angles, is to their difference, so is the sine of the sum of those angles, to the sine of their difference.



Let BCA, ACD, be the two proposed angles, BA and AD their tangents to the radius CA. Take IA = AD, join CI, and draw DE, *de*, IK, *ik*, perpendicular to BC. Then, it is evident, since IA = AD, and CA perpendicular to BD, that CI = CD, ICA = DCA, and therefore BCI = BCA − ACD: also that *ed* is the sine of the sum of the given angles, and *ik* the sine of their difference, to the assumed radius AC.

Now, by reason of the similar triangles BDE, BIK, it will be,

BD (= BA + AD) : BI (= BA − AD) :: DE : IK.

Also, because the triangles CDE, *cde*, are similar, as well as the triangles CIK, *cik*; we have

CD : *cd* :: DE : *de*,

and CI (= CD) : *ci* (= *cd*) :: IK : *ik*;

therefore DE : IK :: *de* : *ik*.

And consequently, BD : BI :: *de* : *ik*;

that is, tan BCA + tan DCA : tan BCA − tan DCA

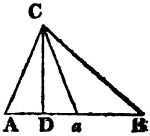
:: sin (BCA + DCA) : sin (BCA − DCA).

Cor. Hence it follows, that the base of a plane triangle BCD, is to the difference of its two segments BA,

AD, as the sine of the whole angle at the vertex, to the sine of the difference of the angles at the base.

Prop. XIX.

19. As the sine of the difference of any two unequal angles, is to the difference of their sines, so is the sum of those sines, to the sine of the sum of the angles.



Let A and B be the two unequal angles; both being bounded on one side by the line AB, and on the other by lines, which, produced, meet at C, and form a rectilinear triangle ACB. Demit CD perpendicularly from C upon AB; make $\mathrm{D}a = \mathrm{DA}$, and join $\mathrm{C}a$. Then we shall have $a\mathrm{B} = \mathrm{DB} - \mathrm{DA}$, $\mathrm{C}a = \mathrm{CA}$, $\mathrm{C}a\mathrm{D} = \mathrm{CAD}$, $\mathrm{C}a\mathrm{B} =$ supp. CAB; and conseq. (ch. i. 19 c) $\sin \mathrm{CAB} = \sin \mathrm{C}a\mathrm{B}$. Also (by the same) $\sin \mathrm{C} = \sin (\mathrm{A} + \mathrm{B})$: and $a\mathrm{CB} = \mathrm{A}a\mathrm{C} - \mathrm{B} = \mathrm{A} - \mathrm{B}$.

Hence, in $\triangle\ \mathrm{ACB}$, $\sin \mathrm{C}$, or $\sin (\mathrm{A} + \mathrm{B}) : \sin \mathrm{A} :: \mathrm{AB} : \mathrm{CB}$.

Also, in $\triangle\ a\mathrm{CB}$, $\sin \mathrm{C}$, or $\sin (\mathrm{A} - \mathrm{B}) : \sin \mathrm{A} :: a\mathrm{B} : \mathrm{CB}$.

Therefore, $a\mathrm{B} : \mathrm{AB} :: \sin (\mathrm{A} - \mathrm{B}) : \sin (\mathrm{A} + \mathrm{B})$.

Again (prop. 14) $\mathrm{CB} : \mathrm{CA} :: \sin \mathrm{A} : \sin \mathrm{B}$.

$\therefore$ comp. and div. $\mathrm{CB} - \mathrm{CA} : \mathrm{CB} + \mathrm{CA} :: \sin \mathrm{A} - \sin \mathrm{B} : \sin \mathrm{A} + \sin \mathrm{B}$.

But, by the inversion of prop. 16,

$$a\mathrm{B} : \mathrm{CB} - \mathrm{CA} :: \mathrm{CB} + \mathrm{CA} : \mathrm{AB}.$$

Therefore, comparing the corresponding antecedents and consequents, in the 3d and 5th analogies, there results,

$$\sin (\mathrm{A} - \mathrm{B}) : \sin \mathrm{A} - \sin \mathrm{B} :: \sin \mathrm{A} + \sin \mathrm{B} : \sin (\mathrm{A} + \mathrm{B}).$$

Cor. If A and B be to each other as $n + 1$ to n, then the last proportion will become

$$\sin \mathrm{A} : \sin (n + 1)\,\mathrm{A} - \sin n\mathrm{A} :: \sin (n + 1)\,\mathrm{A} + \sin n\mathrm{A} : \sin (2n + 1)\,\mathrm{A}.$$

Scholium.

20. Of the propositions in this chapter, some are useful in the solutions of plane triangles, and will find their applications in the next chapter; others are useful

in the construction of tables of natural sines, tangents, &c. To glance, for a moment, at this application, in passing, suppose it were required to find the natural sines to each of the 10 first minutes of the quadrant.

The radius of a circle being 1, the semi-circumference is known to be 3·14159265358979. This being divided successively by 180 and 60, or at once by 10800, gives ·0002908882086657, for the arc of 1 minute. Of so small an arc the sine, chord, and arc, differ almost imperceptibly from the ratio of equality; so that the first ten of the preceding figures, that is, ·0002908882 may be regarded as the sine of 1′; and in fact the sine given in the tables which run to seven places of figures is ·0002909. By chap. i. art. 19, we have, for any arc, $\cos = \sqrt{(1 - \sin^2)}$. This theorem gives, in the present case, $\cos 1' = \cdot 9999999577$. Then, by prop. 12, cor. 2, (A), of this chapter, we shall have

$$2\cos 1' \times \sin 1' - \sin 0' = \sin 2' = \cdot 0005817764$$
$$2\cos 1' \times \sin 2' - \sin 1' = \sin 3' = \cdot 0008726646$$
$$2\cos 1' \times \sin 3' - \sin 2' = \sin 4' = \cdot 0011635526$$
$$2\cos 1' \times \sin 4' - \sin 3' = \sin 5' = \cdot 0014544407$$
$$2\cos 1' \times \sin 5' - \sin 4' = \sin 6' = \cdot 0017453284$$
&c. &c. &c.

Thus may the work be continued to any extent, the whole difficulty consisting in the multiplication of each successive result by the quantity $2\cos 1' = 1\cdot 9999999154$.

Or, the sines of 1′ and 2′ being determined, the work might be continued by the last proposition, thus:

$$\sin 1' : \sin 2' - \sin 1' :: \sin 2' + \sin 1' : \sin 3'$$
$$\sin 2' : \sin 3' - \sin 1' :: \sin 3' + \sin 1' : \sin 4'$$
$$\sin 3' : \sin 4' - \sin 1' :: \sin 4' + \sin 1' : \sin 5'$$
$$\sin 4' : \sin 5' - \sin 1' :: \sin 5' + \sin 1' : \sin 6'$$
&c. &c. &c.

In like manner, the computer might proceed for the sines of degrees, &c. thus:

$$\sin 1^\circ : \sin 2^\circ - \sin 1^\circ :: \sin 2^\circ + \sin 1^\circ : \sin 3^\circ$$
$$\sin 2^\circ : \sin 3^\circ - \sin 1^\circ :: \sin 3^\circ + \sin 1^\circ : \sin 4^\circ$$
$$\sin 3^\circ : \sin 4^\circ - \sin 1^\circ :: \sin 4^\circ + \sin 1^\circ : \sin 5^\circ$$
&c. &c. &c.

To check and verify operations like these, the proportions should be changed at certain stages. Thus,

$$\sin 1^\circ : \sin 3^\circ - \sin 2^\circ :: \sin 3^\circ + \sin 2^\circ : \sin 5^\circ$$
$$\sin 1^\circ : \sin 4^\circ - \sin 3^\circ :: \sin 4^\circ + \sin 3^\circ : \sin 7^\circ$$
$$\sin 4^\circ : \sin 7^\circ - \sin 3^\circ :: \sin 7^\circ + \sin 3^\circ : \sin 10^\circ.$$

The coincidence of the results thus obtained, with the analogous results in the preceding operations, will manifestly establish the correctness of both.

The sines and cosines of the degrees and minutes up to 30°, being determined by these or other processes (some of which will be indicated in the 4th chapter), they may be continued thus:

$$\sin 30^\circ 1' = \cos 1' - \sin 29^\circ 59'$$
$$\sin 30^\circ 2' = \cos 2' - \sin 29^\circ 58'$$
$$\sin 30^\circ 3' = \cos 3' - \sin 29^\circ 57'.$$

And these being continued to 60°, the cosines also become known to 60°; because

$$\cos 30^\circ 1' = \sin 59^\circ 59'$$
$$\cos 30^\circ 2' = \sin 59^\circ 58'.$$

The sines and cosines from 60° to 90°, are deduced from those between 0° and 30°. For

$$\sin 60^\circ 1' = \cos 29^\circ 59'$$
$$\sin 60^\circ 2' = \cos 29^\circ 58'$$
&c. &c. &c.

21. The sines and cosines being found, the *versed sines* are determined by *subtracting* the cosines from radius in arcs *less* than 90°, and by *adding* the cosines to radius in arcs *greater* than 90°.

22. The *tangents* may be found from the sines and cosines. For since $\tan = \frac{\sin}{\cos}$, (chap. i. 19),

$$\text{we have } \tan 1' = \frac{\sin 1'}{\cos 1'} = \cot 89^\circ 59'$$
$$\tan 2' = \frac{\sin 2'}{\cos 2'} = \cot 89^\circ 58'$$
&c. &c. &c.

Above 45° the process may be considerably simplified by the theorem for the tangents of the sums and differences of arcs. For, when the radius is unity, the tangent of 45° is also unity, and tan (A + B) will be denoted thus:

$$\tan (45^\circ + \text{B}) = \frac{1 + \tan \text{B}}{1 - \tan \text{B}}.$$

And this, again, may be still further simplified in practice.

23. The *secants* may readily be found from the tangents by addition. For (prop. 3) sec A = tan A + tan ½ comp A. Or, for the *odd* minutes of the quadrant the secants may be found from the expression $\sec = \frac{1}{\cos}$.

Other methods for all the trigonometrical lines are deduced from the expressions for the sines, tangents, &c. of multiple arcs; but this is not the place to explain them, even if it were requisite to introduce them at large into a cursory outline.

CHAPTER III.

Solutions of the several Cases of Plane Triangles.

1. THERE being in every plane triangle, six things, namely, three sides and three angles, of which some three are given to determine the other three; and the combinations that can be formed out of six quantities, taken three and three being $= \frac{6 \,.\, 5 \,.\, 4}{1 \,.\, 2 \,.\, 3}$ or 20; it might at first sight be imagined that 20 distinct rules would be required in this branch of trigonometry. But though the varieties of data are in truth thus numerous, it will

soon appear, that the number of *cases* which require distinct rules, are very few. Let A, B, C, denote the angles of a plane triangle, and a, b, c, the sides respectively opposite; then the twenty varieties of data will be these, viz.

(3) abc ...	(1) abA ...	(1) abB ...	(2) abC
	(1) acA ...	(2) acB ...	(1) acC
	(2) bcA ...	(1) bcB ...	(1) bcC
(4) ABC ...	(1) ABa ...	(1) ABb ...	(*) ABc
	(1) ACa ...	(*) ACb ...	(1) ACc
	(*) BCa ...	(1) BCb ...	(1) BCc.

Here the varieties which are marked by the figure 1, have this in common, that *a side and its opposite angle* are two of the given parts; and, if it be considered that when two angles of a plane triangle are known, the third is, in fact, given, because it is the supplement of their sum, it will appear that those varieties which are marked with an asterisk fall under this first case: so that the varieties comprehended in this case are *fifteen.* Of the remaining five, three, indicated by the number 2, have this in common, that *two sides and an included angle* are given; they, therefore, constitute a second case. The example marked 3, has *three sides* given, but no angle; this makes a third case: and the remaining variety in which the three angles are given, but no side, would make a fourth case, were it not (see chap. i. 6) that for want of a side among the data, the problem thus expressed is unlimited. Our twenty varieties, therefore, only furnish three distinct cases, to the solution of which we shall now proceed.

CASE I.

2. When a side and its opposite angle are among the given parts.

Here the solution may be obtained by means of chap. ii. prop. 14, where it is demonstrated that the sides of plane triangles are respectively proportional to the sines of their opposite angles.

In practice if a *side* be required, begin the proportion with a sine, and say,

As the sine of the given angle,
To its opposite side;
So is the sine of either of the other angles,
To its opposite side.

If an *angle* be required, begin the proportion with a side, and say,

As one of the given sides,
Is to the sine of its opposite angle;
So is the other given side,
To the sine of its opposite angle.

The third angle becomes known, by taking the sum of the two former from 180°.

Note 1. Since sines are *lines*, there can be no impropriety in comparing them with the sides of triangles; and the rule is better remembered by young mathematicians, than when the sines and sides are compared each to each.

Note 2. It is usually, though not always, best to work the proportions in trigonometry by means of the logarithms, taking the logarithm of the *first* term from the sum of the logarithms of the *second* and *third*, to obtain the logarithm of the *fourth* term. Or, adding the *arithmetical complement* of the logarithm of the first term to the logarithms of the other two, to obtain that of the fourth.

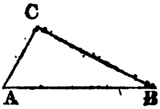
Note 3. It is an excellent plan to accustom the pupil to draw (previously to his commencing the computations in plane trigonometry) not a rough, but a neat and accurate, sketch of the triangle proposed, from the given data, by means of scale, protractor, and compasses. Such construction will enable him at once to trace the peculiarities of the problem, and to detect its ambiguities, if there be any. It will also, if accurately performed, upon a scale of moderate size, give the sides to within their 200th part, and the angles true to within half a degree.

Note 4. The triangle being carefully constructed, and marked by suitable letters of reference, as A and B for example, at the extremities of the base, C at the vertex, then, according to the nature of the problem, write down the requisite proportion in four distinct lines, with the letters of reference to each term, and the given numbers to the three first: against these numbers place their respective logarithms; find the logarithm to the fourth term by the directions in note 2, and ascertain its angular or lineal value, by means of the tables.

⁂ The last three notes are not restricted to the present case, but extend in their application to the usual practice of plane trigonometry.

Example I.

8. In a plane triangle are given two angles equal to 58° 7′, and 22° 37′, respectively, and the side between them 408 yards. Required the remaining angle and sides.



Construction. On an indefinite right line, set off, from a convenient diagonal scale, the distance AB = 408. From the point A draw a right line AC, to make with AB an angle of 58° 7′; and from the point B another line, turned towards the former, and to make with BA an angle of 22° 37′. The intersection C, of these two lines determines the triangle; and the sides AC, BC, measured upon the scale of equal parts from which AB was laid down, are found to be 159 and 351 respectively.

Computation. Two of the angles being known, their sum 80° 44′ taken from 180°, the sum of the angles in a plane triangle, leaves 99° 16′ for the third angle C. This, being an obtuse angle, its sine is to be found in the table by taking that of its supplement 80° 44′, which (chap. i. 19) is the same. Hence,

	Logs.		Logs.
As sin C 99° 16′	9·9942950	As sin C 99° 16′	9·9942950
To AB...408...	2·6106602	To AB...408...	2·6106602
So is sin A 58° 7′	9·9289718	So is sin B 22° 37′	9·5849685
To BC..351·02	2·5453370	To AC..158·98	2·2013337

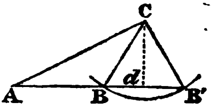
AC and BC, therefore, are 351·02 yards, and 158·98 yards, respectively.

4. In the preceding operation, instead of adding together the logs. of the second and third terms, and subtracting that of the first from their sum, the work has been performed thus:—The right hand figure of the upper line was taken from 10, and each of the other figures from 9, and their several remainders added to the numbers below them in the respective columns. This is easily effected in practice by making those remainders emphatical in adding downwards. Thus, in the operation for BC, we begin at the right hand, and adding downwards say, *ten* and 2 are 12, and 8 are 20; set down 0: carry 2, and *four* are 6, and 1 make 7: *nothing* added to 6 and 7, gives 13; set down 3: carry 1, and *seven* are 8, and 6 are 14, and 9 are 23; set down 3: carry 2, and *five* are 7, and 8 are 15: and so on, to the left hand column, which added in the same way amounts to 12; of which the 2 are put down, and the 10 rejected, to compensate for what has been borrowed in the process by the *arithmetical complement.* This method is very easy in practice, and is found less liable to produce error than that in which the arith. comp. is put down at once from the tables.

Example II.

5. In a plane triangle ABC are given AC = 216; CB = 117; the angle A = 22° 37′; to find the rest.

Construction. Draw an indefinite right line ABB′, from an assumed point A in which draw a line to make the angle A = 22° 37′. Make AC = 216, and from C as a centre with a radius = 117 describe an arc BB′, it will cut the line AB in two points, from each of which drawing lines BC, B′C, there will be formed two triangles ABC, AB′C, each of which answer the conditions of the question. The required lines and angles being measured, give



AB = 117 .... ACB = $22\frac{1}{2}°$ .... ABC = $134\frac{3}{4}°$
AB′ = 282 .... ACB′ = 112° .... AB′C = $45\frac{1}{4}°$.

Computation. To find the angle B:

As BC .... 117 ...... 2·0681859
To sin A .. 22° 37′.... 9·5849685
So is AC .. 216 ...... 2·3344538

To sin B ...... 45° 14′ or 134° 46′.. 9·8512364
Add to each ∠ A 22° 37′ .. 22° 37′

Take the sums 67° 51′ .. 157° 23′
From ...... 180° 0′ .. 180° 0′

Remain ... 112° 9′ or 22° 37′, being ACB′ and ACB.

Here since ACB = BAC, we have AB = BC = 117.

But to find AB, reverse the first two terms of the former proportion, and say,

As sin A ...... 22° 37′ .... 9·5849685
To BC .... 117 .... 2·0681859
So is sin ACB′ .. 112° 9′ .... 9·9667048

To AB′ ...... 281·79 ...... 2·4499222

6. *Remark.* The ambiguity in this and similar examples, does not, as has been often affirmed, *depend* upon the circumstance that an angle and its supplement have the same sine, but solely upon this, that in con-

structing triangles from analogous data, when the side CB has its length between certain limits, that is, between the length of the side AC, and that of the perpendicular *cd* from C on the third side AB, it must *necessarily* cut the indefinite right line ABB′ in two points. Beyond those limits there is no ambiguity; for when CB is proposed to be less than *cd*, the problem is impossible; and when CB exceeds CA, the angle A being all along supposed given, the line ABB′ can only be cut in one point. Hence the practical maxim may be thus expressed:—when CB is proposed less than AC sin A, the data are erroneous; when it is given greater than AC there can be only one triangle; between those limits the problem is ambiguous.

Ex. 3. Given two angles of a plane triangle 22° 37′ and 134° 46′, and the side between them 351. To find the remaining angle and sides.

Ans. Angle 22° 37′; sides 351 and 648.

Ex. 4. Given two sides of a plane triangle 50 and 40 respectively, and the angle opposite to the latter equal to 32°; to determine the triangle.

Ans. If the angle opposite to the side 50 be acute, then is it = 41° 28′, the third angle 106° 32′, and the remaining side 72·36. If the angle opposite to side 50 be obtuse, it is = 138° 32′, and the other angle and side 9° 28′ and 12·415 respectively.

CASE II.

7. When two sides and the included angle are given. The solution is effected by means of chap. ii. prop. 15. Take the given angle from 180°, the remainder will be the sum of the other two angles.

Then say,—As the sum of the given sides,
Is to their difference;
So is the tangent of half the sum of the remaining angles,
To the tangent of half their difference.

Half the difference *added* to half the sum of those angles, gives the greater of them; and *taken* from half the sum, leaves the less*.

All the angles becoming known by this process, the third side is found by the rule in case 1.

Example I.

8. Let there be given in a plane triangle ABC, AC = 450, BC = 540, and the included angle C = 80°; to find the remaining angles and side.

Leaving the construction to be effected by the pupil, I shall proceed to the computation.

BC + AC = 990, BC − AC = 90, 180° − C = 100°.

As BC + AC .... 990 ... 2·9956352
To BC − AC .... 90 .. 1·9542425
So is tan $\frac{1}{2}$(A + B) 50° .. 10·0761865

To tan $\frac{1}{2}$(A − B) .. 6° 11′ 9·0347938

Hence 50° + 6° 11′ = 56° 11′ = A; 50° − 6° 11′ = 43° 49′ = B.

Then, to find the third side AB, say,

As sin B .... 43° 49′ .. 9·8403276
To AC ...... 450 .... 2·6532125
So is sin C .. 80° 0′ .. 9·9933515

To AB ...... 640·08 .. 2·8062364

Example II.

9. The two sides of a triangle are 40 and 32, and the included angle 90°, required the other angles and side.

In this example the operation may be considerably shortened by working *without*, instead of with, the logarithms. For, since the given angle is 90°, the half sum of the remaining angles is 45°, whose natural tangent is unity; and the sum and the difference of the given sides are 72 and 8 respectively. Hence

---

* Let $a + b = s$, and $a - b = d$: then, by addition $2a = s + d$, or $a = \frac{1}{2}s + \frac{1}{2}d$; and, by subtraction $2b = s - d$, or $b = \frac{1}{2}s - \frac{1}{2}d$:

$$\text{As } 72 : 8 :: 1 : \frac{8}{72} = \frac{1}{9} = \cdot 1111111 = \text{nat. tan. } 6^\circ 20'.$$

Therefore $45^\circ + 6^\circ 20' = 51^\circ 20'$, and $45^\circ - 6^\circ 20' = 38^\circ 40'$, are the remaining angles.

And the third side (Euc. i. 47) $= \sqrt{40^2 + 32^2} = \sqrt{2624} = 8\sqrt{41} = 51{\cdot}225$.

⁂ Other methods, still shorter, of solving right angled triangles, will be given before we terminate the present chapter.

Ex. 3. Given two sides of a plane triangle 1686, and 960, and their included angle 128° 4′; to find the rest.

Ans. Angles 33° 35′, 18° 21′, side 2400.

Case III.

10. **When the three sides of a plane triangle are given, to find the angles.**

1st *Method.* Assume the longest of the three sides as base, then say, conformably with chap. ii. prop. 16.

As the base,
To the sum of the two other sides;
So is the difference of those sides,
To the difference of the segments of the base.

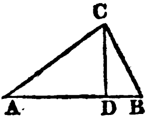
Half the base added to the said difference, gives the greater segment, and made less by it gives the less; and thus, by means of the perpendicular from the vertical angle, divides the original triangle into two, each of which falls under the first case.

2d *Method.* Find any one of the angles by means of prop. 17, of the preceding chapter; and the remaining angles either by a repetition of the same rule, or by the relation of sides to the sines of their opposite angles.

Example I.

11. The three sides of a triangle are 40, 34, and 25. Required the angles.

By Rule I.



As AB : AC + BC :: AC − BC : AD − BD;

that is, $40 : 59 :: 9 : \frac{59 \times 9}{40} = 13{\cdot}275.$

Then $\frac{40 + 13{\cdot}275}{2} = 26{\cdot}6375 =$ AD:

and $\frac{40 - 13{\cdot}275}{2} = 13{\cdot}3625 =$ DB.

As AC	34	1·5314789	As CB	25	1·3979400
To sin D	90° ..	10·0000000	To sin D ...	90° ..	10·0000000
So is AD ..	26·6375	1·4254935	So is DB ..	13·3625	1·1258878
To sin ACD ..	51° 35′	9·8940146	To sin BCD ..	32° 18′	9·7279478

Hence 90° − 51° 35′ = 38° 25′ = A; 90° − 32° 18′ = 57° 42′ = B;
and 51° 35′ + 32° 18′ = 83° 53′ = ACB.

By Rule II.

$AC^2 = 1156$, $BC^2 = 625$, $AB^2 = 1600$, $CB \cdot CA = 850$.

From log $AC^2 + BC^2 - AB^2 =$ log 181 = 2·2576786
Take log $2CB \cdot CA =$ log 1700 = 3·2304489

Rem. + 10 (in the index) = log cos C = 83° 53′ = 9·0272297

The other two angles may be found by case 1.

Ex. 2. When the sides of a plane triangle are 408, 351, and 159, what are the angles?

Ans. 99° 16′, 58° 7′, and 22° 37′.

Ex. 3. When the sides are 4, 5, and 6, what are the angles?

Ans. 41° 24′ 35″, 55° 46′ 16″, and 82° 49′ 9″.

RIGHT ANGLED PLANE TRIANGLES.

12. 1. Right angled triangles may, as well as others, be solved by means of the rule to the respective case under which any specified example falls: and it will then be found, since a right angle is always one of the data, that the rule usually becomes simplified in its application; as appeared in the solution of the second example to case 2.

2. When two of the sides are given, the third may be found by means of the property demonstrated in Euc. i. 47. Thus,

$$\text{Hypoth.} = \sqrt{(\text{base}^2 + \text{perp.}^2.)}$$

$$\text{Base} = \sqrt{(\text{hyp.}^2 - \text{perp.}^2)} = \sqrt{(\text{hyp.} + \text{perp.}).(\text{hyp.} - \text{perp.})}$$

$$\text{Perp.} = \sqrt{(\text{hyp.}^2 - \text{base}^2)} = \sqrt{(\text{hyp.} + \text{base}).(\text{hyp.} - \text{base})}.$$

3. There is another method for right angled triangles, known by the phrase *making any side radius;* which is this.

" *To find a side.* Call any one of the sides radius, and write upon it the word *radius;* observe whether the other sides become sines, tangents, or secants, and write those words upon them accordingly. Call the word written upon each side the *name* of each side: then say,

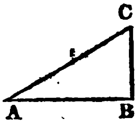
" As the *name* of the given side,
" Is to the given side;
" So is the *name* of the required side,
" To the required side."

" *To find an angle.* Call either of the *given* sides radius, and write upon it the word *radius;* observe whether the other sides become sines, tangents, or secants, and write those words on them accordingly. Call the word written upon each side the *name* of that side. Then say,

" As the side made radius,
" Is to radius;
" So is the other given side,
" To the *name* of that side:

which determines the opposite angle."

13. When the numbers which measure the sides of the triangle, are either under 12, or resolvable into factors which are each less than 12, the solution may be obtained, conformably with this rule, easier without logarithms than with them. For,



Let ABC be a right angled triangle, in which AB the base is assumed to be radius; BC is the tangent of A, and AC its secant, to that radius; or, dividing each of these by the base, we shall have the tangent and secant of A, respectively, to radius 1. Tracing in like manner, the consequences of assuming BC, and AC, each for radius, we shall readily obtain these expressions.

1. $\frac{\text{perp.}}{\text{base}}$ = tan angle at base.

2. $\frac{\text{base}}{\text{perp.}}$ = tan angle at vertex.

3. $\frac{\text{hyp.}}{\text{base}}$ = sec angle at base.

4. $\frac{\text{hyp.}}{\text{perp.}}$ = sec angle at vertex.

5. $\frac{\text{perp.}}{\text{hyp.}}$ = sin angle at base.

6. $\frac{\text{base}}{\text{hyp.}}$ = sin angle at vertex.

Example I.

14. In a right angled triangle are given, the hypothenuse and the base, 25 and 24 respectively; to find the rest.

$\text{Perp.} = \sqrt{(\text{hyp.}^2 - \text{base}^2)} = \sqrt{(25+24).(25-24)} = 7$, $\frac{\text{perp.}}{\text{base}} = \frac{7}{24} = \frac{1{\cdot}75}{6} = {\cdot}2916666 = \tan 16^\circ\ 15'\ 37''$, angle at the base; whence $90^\circ - 16^\circ\ 15'\ 37'' = 73^\circ\ 44'\ 23''$, angle at vertex.

Or, $\frac{\text{base}}{\text{perp.}} = \frac{24}{7} = 3{\cdot}4\dot{2}8571\dot{4} = \tan 73^\circ 44' 23''$, ang. at vert.

Or, thus, by the secants.

$$\frac{\text{hyp.}}{\text{base}} = \frac{25}{24} = \frac{6{\cdot}25}{6} = 1{\cdot}0416666 = \sec 16^\circ\ 15'\ 37'',$$

$$\frac{\text{hyp.}}{\text{perp.}} = \frac{25}{7} = 3{\cdot}5714285 = \sec 73^\circ\ 44'\ 23''.$$

Or, thus, by the sines.

$$\frac{\text{perp.}}{\text{hyp.}} = \frac{7}{25} = \frac{28}{100} = 0{\cdot}28 = \sin 16^\circ\ 15'\ 37'',$$

$$\frac{\text{base}}{\text{hyp.}} = \frac{24}{25} = \frac{96}{100} = 0{\cdot}96 = \sin 73^\circ\ 44'\ 23''.$$

Ex. 2. In a right angled triangle, given a leg and its opposite angle 384, and 53° 8′ respectively; to find the other leg and the hypothenuse.

Ans. Leg 280, hypothenuse 480.

Ex. 3. In a right angled triangle are given the base 195, its adjacent angle 47° 55′; to find the rest.

Ans. Perp. 216, hyp. 291, vert. ang. 42° 5′.

CHAPTER IV.

Plane Trigonometry considered Analytically.

1. IN the preceding chapters the investigation of trigonometrical properties has been conducted *geometrically;* the various relations of the sines, cosines, tangents, &c. of arcs and angles, whether depending upon triangles or not, being deduced immediately from the figures to which the several enquiries were referred. This method carries conviction at every step; and by keeping the objects of enquiry constantly before the

eye of the student, serves admirably to guard him against the admission of error; and is, therefore, well fitted for adoption at the commencement of this branch of science. But of late years, another method, first introduced by Euler, has been generally employed by the continental mathematicians, and very frequently by those of England. It is *analytical.* The nature and mutual relation of the lineo-angular quantities, sines, tangents, &c. being defined by a few obvious and simple equations, every other theorem and formula that is likely to be of use, is deduced with great facility by the mere reduction and transformation of the original equations. This method serves greatly to shorten almost all trigonometrical investigations, except a few which lie at the foundation of the science: and admitting of an extension to which the geometrical method cannot lay claim, at the same time that it proceeds to some of its most important results with great rapidity; no treatise on trigonometry can be complete that does not assign to this manner of handling the subject a considerable place.

2. Previously to the student's entrance upon this department of enquiry, it will be expedient for him to trace the mutations which the principal intermediary or lineo-angular quantities undergo, when they relate to arcs found in different quadrants of the same circle.

To this end let him draw afresh and lay before him, the diagram given at chap. i. art. 18, and first trace the mutations of the sines and the cosines; keeping in mind this general principle, that *every variable algebraic quantity changes its sign after it becomes* 0.

3. Let the arc be supposed to commence at A, and to increase in the direction ABEA′B. As the arc augments through the first quadrant AE, the sine augments till it becomes equal to CE, the radius; passing from 90° to 180° through the quadrant EA′, the sine diminishes, but is to be regarded as *positive* till the arc becomes AEA′ or 180°. In that state of the arc the value of the sine

is obviously 0. Passing on to the third quadrant, as when the arc is AEA′B′, the sine D′B′ is directed contrarily to what it was in the first two quadrants, and is then to be regarded as *negative*. In this state it continues to increase through the third quadrant, at the end E′ of which it is again equal to radius. From thence the negative value of the sine diminishes, till at the end A of the fourth arc, it again passes through *zero*, and the sine becomes *positive* in the *fifth* quadrant; as it obviously ought to do, since the fifth quadrant is coincident with the first.

In the 1st and 2d quadrants, then, the sines are +,
In the 3d and 4th, they are −.

If the arc were taken negatively, its sine would, for like reasons, be *negative* in the first half, *positive* in the second half, of the circle.

4. The *cosine* of an arc is equal to radius at the point A, where the arc is evanescent. From thence it diminishes while the arc increases through the first quadrant; at E the cosine is 0. Then it changes its sign and continues *negative* through the arc EA′E′, that is, from 90° to 270°. At E′ the cosine is again 0; and, of course, from E′ to A, through the fourth quadrant, it is again *positive*.

The rule is the same for a negative arc.

5. With regard to the signs of the *tangents*, it is evident, since $\tan = \frac{\sin}{\cos}$ (chap. i. 19), that when the signs of the sine and cosine are both *alike*, whether positive or negative, the sign of the tangent is *positive*; and when those signs are *unlike*, the sign of the tangent is *negative*. The tangent, therefore, is *positive* in the *first* quadrant, the sine and cosine being then both positive: it is *negative* in the *second* quadrant, the cosine being then negative though the sine remains positive; it becomes *positive* again in the *third* quadrant, the sine and cosine being then both negative: finally, it is *negative*

in the *fourth* quadrant, since the sine is then negative and the cosine positive. The *tangent*, therefore, changes its sign at the end of every quadrant, where it passes alternately through nothing and infinity; these being, indeed, the algebraic indications of the changes of sign.

If the arc is regarded as negative, the rule for the tangents becomes inverted; the tangent being then *negative* in the *first* and *third*, *positive* in the *second* and *fourth* arcs.

6. For the *cotangents* the rule of the signs is the same as for the tangents: this is evident, because $\tan = \frac{1}{\cot}$, and $\cot = \frac{1}{\tan}$.

7. So again, the rule for the *secants* is the same as for the *cosines*; because $\sec = \frac{1}{\cos}$.

8. The rule for the *cosecants* is, also, the same as for the sines; because $\text{cosec} = \frac{1}{\sin}$.

9. Proceeding in this way, a second, a third, &c. time over the circumference of the circle, like mutations would occur. The results of the whole may be thus tabulated:—

quadrants	sin	cos	tan	cot	sec	cosec
1st..5th..9th..13th	+	+	+	+	+	+
2d..6th..10th..14th	+	−	−	−	−	+
3d..7th..11th..15th	−	−	+	+	−	−
4th..8th..12th..16th	−	+	−	−	+	−

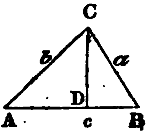
(A.)

10. Let it be remembered that when the tangent passes through infinity, the *secant* does, and for the same reason, viz. because they in that case could only limit each other at an infinite distance. The principal changes, then, in point of magnitude, may be easily traced, and tabulated as below; where the sign ∞ denotes *infinity*.

	0°	90°	180°	270°	360°
Sin	0	R	0	−R	0
tan	0	∞	0	−∞	0
sec	R	∞	−R	−∞	R
cos	R	0	−R	0	R
cot	∞	0	−∞	0	∞
cosec	∞	R	−∞	R	∞

(B.)

11. These particulars premised, the properties and relations of the sines, tangents, &c. of combined and multiple arcs, as well as rules for the solutions of triangles, may be investigated analytically.



Let ABC, in the annexed figure, be any plane triangle, C the vertical angle, CD a perpendicular let fall from it upon the base or base produced, and let a, b, and c, denote the sides respectively opposite to the angles A, B, and C.

Then, since AC $= b$, AD is the cosine of A to that radius; consequently, when radius is unity we have AD $= b \cos$ A. In like manner BD $= a \cos$ B. Therefore, AD + BD = AB $= c = a \cos B + b \cos A$. If one of the angles, as A, were obtuse, the result would, notwithstanding, be the same; because, while on the one hand cos A would be negative, AD, lying on the contrary side of A to what it does in the figure, must be deducted from BD to leave AB, and a negative quantity subtracted, is equivalent to a positive quantity added. By letting fall perpendiculars from the angles A and B, upon the opposite sides, or their continuations, precisely analogous results will be obtained. They may be placed together, thus:—

$$\left.\begin{array}{l} a = b \cos C + c \cos B \\ b = a \cos C + c \cos A \\ c = a \cos B + b \cos A \end{array}\right\} (1.)$$

12. If BC, or a, be regarded as radius, CD will be the sine of the angle B to that radius; therefore, to the radius unity, CD will be $= a \sin$ B. So again, for a like

reason, CD $= b \sin$ A. Consequently, $a \sin$ B $= b \sin$ A, or $\frac{a}{b} = \frac{\sin \text{A}}{\sin \text{B}}$. In a similar manner, we may obtain $\frac{a}{c} = \frac{\sin \text{A}}{\sin \text{C}}$, and $\frac{b}{c} = \frac{\sin \text{B}}{\sin \text{C}}$. Or, changing the denominators, the relations of all the six quantities may be thus expressed,

$$\frac{\sin \text{A}}{a} = \frac{\sin \text{B}}{b} = \frac{\sin \text{C}}{c} \ldots\ldots (2.)$$

These equations are, manifestly, of similar import with chap. ii. prop. 14.

13. From the last equations we have,

$$a = c\frac{\sin \text{A}}{\sin \text{C}} \ldots\ldots b = c\frac{\sin \text{B}}{\sin \text{C}}.$$

Substituting these values of a and b for them in the equation $c = a \cos \text{B} + b \cos \text{A}$, and multiplying the equation so transformed, by $\frac{\sin \text{C}}{c}$, it will become

$$\sin \text{C} = \sin \text{A} \cos \text{B} + \sin \text{B} \cos \text{A}.$$

Now, since in every plane triangle, the sum of the three angles is equal to two right angles, A + B = supplement of C; and, since an angle and its supplement have the same sine, it follows that $\sin (\text{A} + \text{B}) = \sin \text{C}$; whence

$$\sin (\text{A} + \text{B}) = \sin \text{A} \cos \text{B} + \sin \text{B} \cos \text{A}.$$

14. If in this equation B be regarded as subtractive, then will sin B obviously be subtractive also; but cos B will not change its sign, because it will still continue to be estimated in the same direction on the same radius The equation will, therefore, become

$$\sin (\text{A} - \text{B}) = \sin \text{A} \cos \text{B} - \sin \text{B} \cos \text{A}.$$

15. Conceive B′ to be the complement of B, and $\frac{1}{4}$ ○ to be the quarter of the circumference, or the measure of a right angle: then will $\text{B}' = \frac{1}{4} \bigcirc - \text{B}$, $\sin \text{B}' = \cos \text{B}$, and $\cos \text{B}' = \sin \text{B}$. But, by the preceding article, $\sin (\text{A} - \text{B}') = \sin \text{A} \cos \text{B}' - \sin \text{B}' \cos \text{A}$. Substituting for

$\sin B'$, $\cos B'$, their values, there will result $\sin(A - B') = \sin A \sin B - \cos A \cos B$. Hence, because $B' = \frac{1}{4}\bigcirc - B$, we have $\sin(A - B') = \sin(A + B - \frac{1}{4}\bigcirc) = \sin[(A + B) - \frac{1}{4}\bigcirc] = -\sin[\frac{1}{4}\bigcirc - (A + B) = -\cos(A + B)]$. This value of $\sin(A - B')$ being substituted for it in the equation above, it becomes

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

16. If B in this latter equation be made subtractive, $\sin B$ will become $-\sin B$, while $\cos B$ will not change (art. 3, 4). The equation will consequently be transformed into this: viz.

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

If A and B be regarded as *arcs* instead of the angles which they measure, the results will be equally conclusive and correct. They may be expressed generally for the sines and cosines of the sums or differences of any two arcs or angles, by these two equations, viz.

$$\left.\begin{aligned}\sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A\\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B\end{aligned}\right\}\ (\text{C.})$$

17. The actual value of the sine, cosine, &c. depends, obviously, not only upon the magnitude of the arc, but also upon that of the assumed radius. In the preceding investigation we have supposed it to be unity. If we wish to make the above or any other formulæ, applicable to cases where the radius has another value R, we have only to substitute in the expression, $\frac{\sin A}{R}$ for $\sin A$, $\frac{\cos A}{R}$ for $\cos A$, $\frac{\tan A}{R}$ for $\tan A$, and so on. Or, generally, we must so distribute the several powers of R, as to make all the terms *homogeneous*, as to the *number* of lines multiplied: this is effected by multiplying each term by such a power of R as shall make it of the same dimension, as that term in the equation which has the highest dimension. Thus the expression,

$$\sin 3A = 3 \sin A - 4 \sin^3 A \ (\text{rad} = 1)$$

when radius is assumed $= R$, becomes

$$\text{R}^2 \sin 3\text{A} = \text{R}^2\, 3 \sin \text{A} - 4 \sin^3 \text{A},$$

$$\text{or } \sin 3\text{A} = \frac{3\text{R}^2 \sin \text{A} - 4 \sin^3 \text{A}}{\text{R}^2}.$$

18. Supposing that in formulæ (C) A = B, and taking the superior sign, we shall have

$$\left.\begin{array}{l} \sin 2\text{A} = 2 \sin \text{A} \cos \text{A} = 2 \sin \text{A} \sqrt{(1 - \sin^2 \text{A})} \\ \cos 2\text{A} = \cos^2 \text{A} - \sin^2 \text{A} = 2 \cos^2 \text{A} - 1 \end{array}\right\} (\text{D})$$

by substituting $1 - \cos^2 \text{A}$ for $\sin^2 \text{A}$, and $1 - \sin^2 \text{A}$ for $\cos^2 \text{A}$.

So, again, by supposing B to be successively equal to 2A, 3A, 4A, &c. we shall find

$$\left.\begin{array}{l} \sin 3\text{A} = \sin \text{A} \cos 2\text{A} + \cos \text{A} \sin 2\text{A} \\ \sin 4\text{A} = \sin \text{A} \cos 3\text{A} + \cos \text{A} \sin 3\text{A} \\ \sin 5\text{A} = \sin \text{A} \cos 4\text{A} + \cos \text{A} \sin 4\text{A} \end{array}\right\} (\text{E.})$$

Or, for the successive cosines,

$$\left.\begin{array}{l} \cos 3\text{A} = \cos \text{A} \cos 2\text{A} - \sin \text{A} \sin 2\text{A} \\ \cos 4\text{A} = \cos \text{A} \cos 3\text{A} - \sin \text{A} \sin 3\text{A} \\ \cos 5\text{A} = \cos \text{A} \cos 4\text{A} - \sin \text{A} \sin 4\text{A} \end{array}\right\} (\text{F.})$$

19. If in the above expressions for the several multiples of the sines, we introduce for the cosines their values in terms of the sines, and in those for the multiples of the cosines the values of the sines in terms of the cosines, the following expressions will be obtained, in which each quantity is expressed in terms of its own kind.

$$\left.\begin{array}{ll} \sin \text{A} = & \text{s} \\ \sin 2\text{A} = & 2\text{s} \sqrt{(1 - \text{s}^2)} \\ \sin 3\text{A} = & 3\text{s} - 4\text{s}^3 \\ \sin 4\text{A} = & (4\text{s} - 8\text{s}^3) \sqrt{(1 - \text{s}^2)} \\ \sin 5\text{A} = & 5\text{s} - 20\text{s}^3 + 16\text{s}^5 \\ \sin 6\text{A} = & (6\text{s} - 32\text{s}^3 + 32\text{s}^5) \sqrt{(1 - \text{s}^2)} \end{array}\right\} (\text{G.})$$

$$\&\text{c.} \qquad \&\text{c.} \qquad \&\text{c.}$$

$$\left.\begin{array}{ll} \cos \text{A} = & \text{c} \\ \cos 2\text{A} = & 2\text{c}^2 - 1 \\ \cos 3\text{A} = & 4\text{c}^3 - 3\text{c} \\ \cos 4\text{A} = & 8\text{c}^4 - 8\text{c}^2 + 1 \\ \cos 5\text{A} = & 16\text{c}^5 - 20\text{c}^3 + 5\text{c} \\ \cos 6\text{A} = & 32\text{c}^6 - 48\text{c}^4 + 18\text{c}^2 - 1 \end{array}\right\} (\text{H.})$$

20. Other useful expressions for the sines and cosines of multiple arcs, may be found thus:

Take the sum of the expanded expressions for $\sin(B + A)$ and $\sin(B - A)$; that is,

add........ $\sin(B + A) = \sin B \cos A + \cos B \sin A$
to .... $\sin(B - A) = \sin B \cos A - \cos B \sin A$
there results, $\sin(B + A) + \sin(B - A) = 2 \cos A \sin B$.

So again, the sum of the expressions for $\cos(B + A)$ and $\cos(B - A)$, is

$$\cos(B + A) + \cos(B - A) = 2 \cos A \cos B.$$

Whence, $\cos(B + A) = 2 \cos A \cos B - \cos(B - A)$.

Substituting in these expressions for the sine and cosine of $B + A$, the successive values of A, $2A$, $3A$, &c. instead of B, we shall have,

$$\left.\begin{array}{l}\sin 2A = 2 \cos A \sin A \\ \sin 3A = 2 \cos A \sin 2A - \sin A \\ \sin 4A = 2 \cos A \sin 3A - \sin 2A \\ \sin nA = 2 \cos A \sin (n - 1) A - \sin (n - 2) A\end{array}\right\} (\mathrm{I.})$$

$$\left.\begin{array}{l}\cos 2A = 2 \cos A \cos A - \cos 0 \; (= 1) \\ \cos 3A = 2 \cos A \cos 2A - \cos A \\ \cos 4A = 2 \cos A \cos 3A - \cos 2A \\ \cos nA = 2 \cos A \cos (n - 1) A - \cos (n - 2) A\end{array}\right\} (\mathrm{K.})$$

21. If the cosine of A be represented by a particular binomial, the formula (K) will be transformed into a class of very elegant and curious theorems: thus

$$\text{make } 2 \cos A = z + \frac{1}{z};$$

$$\text{then } 2 \cos 2A = 2\,(2 \cos^2 A - 1) = 2\left[\tfrac{1}{4}\left(z + \frac{1}{z}\right)^2 - 1\right]$$

$$= z^2 + \frac{1}{z^2}.$$

A like substitution in the forms for $2 \cos 3A$, $2 \cos 4A$, &c. will give

$$\left.\begin{array}{l} 2\cos 3\text{A} = z^3 + \dfrac{1}{z^3} \\ 2\cos 4\text{A} = z^4 + \dfrac{1}{z^4} \\ 2\cos 5\text{A} = z^5 + \dfrac{1}{z^5} \\ 2\cos n\text{A} = z^n + \dfrac{1}{z^n} \end{array}\right\} (\text{L.})^*$$

22. By an inverted process, the sine and cosine of a single arc may be inferred from those of a double arc; or, which comes to the same, those of a half arc from those of a whole arc.

Let $x - z = \sqrt{(1 - \cos 2\text{A})}$, or $x^2 - 2xz + z^2 = 1 - \cos 2\text{A}$, and assume $x^2 + z^2 = 1$; then $2xz = \cos 2\text{A}$; and, exterminating z, $x^2 + \dfrac{\cos^2 2\text{A}}{4x^2} = 1$.

Hence $x^4 - x^2 + \frac{1}{4} = \frac{1}{4}(1 - \cos^2 2\text{A}) = \frac{1}{4}\sin^2 2\text{A}$,

$\therefore x^2 = \frac{1}{2} \pm \frac{1}{2}\sin 2\text{A}$, and $x = \frac{1}{2}\sqrt{(1 \pm \sin 2\text{A})}$

and $z^2 = \frac{1}{2} \mp \frac{1}{2}\sin 2\text{A}$, and $z = \frac{1}{2}\sqrt{(1 \mp \sin 2\text{A})}$.

Hence, $\sin \text{A} = \frac{1}{2}\sqrt{(1 + \sin 2\text{A})} - \frac{1}{2}\sqrt{(1 - \sin 2\text{A})}$

$\cos \text{A} = \frac{1}{2}\sqrt{(1 + \sin 2\text{A})} + \frac{1}{2}\sqrt{(1 - \sin 2\text{A})}$.

Or, for the half arc,

$$\left.\begin{array}{l} \sin \frac{1}{2}\text{A} = \frac{1}{2}\sqrt{(1 + \sin \text{A})} - \frac{1}{2}\sqrt{(1 - \sin \text{A})} \\ \cos \frac{1}{2}\text{A} = \frac{1}{2}\sqrt{(1 + \sin \text{A})} + \frac{1}{2}\sqrt{(1 - \sin \text{A})} \end{array}\right\} (\text{M.})$$

---

* This remarkable class of formulæ may be of use in the summation of series. Thus, $\cos \text{A} + \cos 2\text{A} + \cos 3\text{A} + \cos 4\text{A} + \&\text{c}. .. \cos n\text{A}$, is equal to the sum of the two series $\frac{1}{2}(z + z^2 + z^3 + \&\text{c}... z^n)$, and $\frac{1}{2}\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \&\text{c}.. \frac{1}{z^n}\right)$: this is equal $\left(\dfrac{z^{n+1} - z}{z - 1} + \dfrac{z^n - 1}{z^n(z - 1)}\right)$; which, after due reduction, comes at last to $\left(\sin \frac{n}{2}\text{A} \cos \frac{n+1}{2}\text{A}\right) \div \sin \frac{1}{2}\text{A}$. This theorem employed to determine the sum of all the natural cosines of every minute in the quadrant, gives $\sin 45° \times \cos 45° \, 0' \, 30'' \div \sin 30'' = 3437{\cdot}2470374$.

23. Otherwise, in order to obtain expressions for the sine and cosine of a half arc, take

$$1 = \cos^2 A + \sin^2 A$$

$$\cos 2A = \cos^2 A - \sin^2 A \text{ (see formulæ D)},$$

the sum and difference of these respectively, give

$$2\cos^2 A = 1 + \cos 2A, \text{ and } 2\sin^2 A = 1 - \cos 2A.$$

Changing A and 2A into $\frac{1}{2}$A and A, we shall have

$$2\cos^2 \tfrac{1}{2}A = 1 + \cos A, \text{ and } 2\sin^2 \tfrac{1}{2}A = 1 - \cos A,$$

$$\left.\begin{array}{l} \therefore \cos \frac{1}{2}A = \sqrt{\dfrac{1+\cos A}{2}};\ \sin \frac{1}{2}A = \sqrt{\dfrac{1-\cos A}{2}} \\ \text{or } \cos \frac{1}{2}A = \frac{1}{2}\sqrt{(2+2\cos A)} \\ \quad \sin \frac{1}{2}A = \frac{1}{2}\sqrt{(2-2\cos A)} \end{array}\right\} \text{(N.)}$$

Suppose, for example, A = 60°, then $\cos A = \sin 30° = \frac{1}{2}$,

and $\cos 30° = \frac{1}{2}\sqrt{(2+1)} = \frac{1}{2}\sqrt{3} = \cdot 8660254$,

$\sin 30° = \frac{1}{2}\sqrt{(2-1)} = \frac{1}{2}$, as it ought to be:

$\cos 15° = \frac{1}{2}\sqrt{(2+\sqrt{3})} = \frac{1}{4}(\sqrt{6}+\sqrt{2}) = \cdot 9659258$

$\sin 15° = \frac{1}{2}\sqrt{(2-\sqrt{3})} = \frac{1}{4}(\sqrt{6}-\sqrt{2}) = \cdot 2588190$:

and so on, by continual bisections, as low as we please.

24. If the formulæ (C) be divided one by the other, there will result,

$$\frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A \cos B \pm \cos B \sin A}{\cos A \cos B \mp \sin A \sin B}.$$

Here dividing the two terms of the second member of the equation by cos A cos B, and recollecting that $\tan A = \frac{\sin A}{\cos A}$, we shall obtain,

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \dots \text{(O.)}$$

If A in this formula be = 45°, we shall have tan A = 1, and consequently,

$$\tan(45° \pm B) = \frac{1 \pm \tan B}{1 \mp \tan B} \dots \text{(P.)}$$

If A = B, the formula (O) will give for the double arc,

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \dots \text{(Q.)}$$

While, on the other hand, dividing the first values of $\sin \frac{1}{2}A$, $\cos \frac{1}{2}A$ (formula N) one by the other, there will result,

$$\tan \tfrac{1}{2}A = \sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{1-\cos A}{\sin A} \ldots\ldots (\text{R.})$$

$$\text{Thus } \tan 30^\circ = \frac{1-\cos 60^\circ}{\sin 60^\circ} = \frac{1-\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \tfrac{1}{3}\sqrt{3};$$

$$\text{and } \tan 15^\circ = \frac{1-\cos 30^\circ}{\sin 30^\circ} = \frac{1-\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = 2-\sqrt{3};$$

and so on, to any proposed extent.

25. Pursuing an analogous process to that by which we obtained $\tan (A \pm B)$, the following may be deduced from formulæ (C): viz.

$$\cot (A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A} \ldots\ldots (\text{O}'.)$$

$$\left.\begin{array}{l}
\dfrac{\sin (A+B)}{\sin (A-B)} = \dfrac{\cot B + \cot A}{\cot B - \cot A} = \dfrac{\tan A + \tan B}{\tan A - \tan B} \\
\dfrac{\sin (A \pm B)}{\cos (A \mp B)} = \dfrac{\cot B \pm \cot A}{\pm 1 + \cot A \cot B} = \dfrac{\tan A \pm \tan B}{1 \pm \tan A \tan B} \\
\dfrac{\cos (A+B)}{\cos (A-B)} = \dfrac{\cot B - \tan A}{\cot B + \tan A} = \dfrac{1 - \tan A \tan B}{1 + \tan A \tan B}
\end{array}\right\} (\text{S.})$$

26. Taking again the equations (C) for the sine of the sum and the difference, there results,

by add. $\sin (A+B) + \sin (A-B) = 2 \sin A \cos B$
by sub. $\sin (A+B) - \sin (A-B) = 2 \sin B \cos A$.

The first of these formulæ divided by the second gives

$$\frac{\sin (A+B) + \sin (A-B)}{\sin (A+B) - \sin (A-B)} = \frac{\sin A}{\cos A} \cdot \frac{\cos B}{\sin B} = \frac{\tan A}{\tan B}.$$

If, therefore, we suppose, $A + B = A'$, $A - B = B'$, whence $A = \frac{1}{2}(A' + B')$, $B = \frac{1}{2}(A' - B')$, we shall have

$$\frac{\sin A' + \sin B'}{\sin A' - \sin B'} = \frac{\tan \frac{1}{2}(A' + B')}{\tan \frac{1}{2}(A' - B')} \ldots\ldots (\text{T.})$$

But, it has been seen (formula 2) that

$\dfrac{\sin A'}{\sin B'} = \dfrac{a}{b}$, whence it follows that $\dfrac{\sin A' + \sin B'}{\sin A' - \sin B'} = \dfrac{a+b}{a-b}$.

The preceding equation, therefore, establishes the truth of the following, viz.

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A'+B')}{\tan\frac{1}{2}(A-B')}\ldots\ldots(3.)$$

27. Similar computations with the equations (C), combined 2 and 2, would produce a great variety of other formulæ. From these, however, since the ingenious student may investigate them nearly as rapidly as he can write them down, we shall only select the following for insertion here.

$$\left.\begin{aligned}\sin A+\sin B&=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\\ \sin A-\sin B&=2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B)\\ \cos A+\cos B&=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\\ \cos B-\cos A&=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)\end{aligned}\right\}(U.)$$

Making $A=90°$ in the first two, and $B=0$, in the others, there result,

$$1+\sin B=2\sin\left(45°+\tfrac{1}{2}B\right)\cos\left(45°-\tfrac{1}{2}B\right)=2\sin^2\left(45°+\tfrac{1}{2}B\right)$$

$$1-\sin B=2\sin^2\left(45°-\tfrac{1}{2}B\right)=2\cos^2\left(45°+\tfrac{1}{2}B\right)$$

$$\left.\begin{aligned}1+\cos A&=2\cos^2\tfrac{1}{2}A\\ 1-\cos A&=2\sin^2\tfrac{1}{2}A\end{aligned}\right\}(U'.)$$

$$\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A+B);\quad\frac{\sin A-\sin B}{\cos B-\cos A}=\cot\tfrac{1}{2}(A+B)$$

$$\frac{\sin A+\sin B}{\cos B-\cos A}=\cot\tfrac{1}{2}(A-B);\quad\frac{\sin A-\sin B}{\cos A+\cos B}=\tan\tfrac{1}{2}(A-B)$$

$$\frac{\cos A+\cos B}{\cos A-\cos B}=-\cot\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B)$$

$$\frac{1+\sin B}{1-\sin B}=\tan^2\left(45°+\tfrac{1}{2}B\right);\quad\frac{1+\cos A}{1-\cos A}=\cot^2\tfrac{1}{2}A$$

$$\frac{1+\sin B}{1+\cos A}=\frac{\sin^2\left(45°+\frac{1}{2}B\right)}{\cos^2\frac{1}{2}A};\quad\frac{1-\sin B}{1-\cos B}=\frac{\sin^2\left(45°-\frac{1}{2}B\right)}{\sin^2\frac{1}{2}B}.$$

28. Some useful theorems in the computation of sines, cosines, &c. result from the introduction of the *imaginary* quantities that arise from the solution of the equations $A^2+1=0$, and $B^2+1=0$: so that, though these roots, $A=\pm\sqrt{-1}$, and $B=\pm\sqrt{-1}$, are in-

dications of an absurd supposition, they still serve as convenient instruments of investigation. Their utility must, therefore, be briefly shewn.

The expression $\sin^2 A + \cos^2 A = R$ or 1, is resolvable into the imaginary factors,

$$(\cos A + \sin A \sqrt{-1}) . (\cos A - \sin A \sqrt{-1}) = 1.$$

In like manner, introducing another arc B, and employing imaginary factors, we shall have $(\cos A + \sin A \sqrt{-1}) . (\cos B + \sin B \sqrt{-1}) = \cos A \cos B - \sin A \sin B + \sqrt{-1} (\cos A \sin B + \sin A \cos B) = \cos (A + B) + \sin (A + B) \sqrt{-1}$, [by formula C]; also $(\cos A - \sin A \sqrt{-1}) . (\cos B - \sin B \sqrt{-1}) = \cos (A + B) - \sin (A + B) \sqrt{-1}$. By a like calculation it will be seen that $(\cos A \pm \sin A \sqrt{-1}) . (\cos B \pm \sin B \sqrt{-1}) . (\cos C \pm \sin C \sqrt{-1}) = \cos (A + B + C) \pm \sin (A + B + C) \sqrt{-1}$. It appears, therefore, that the multiplication of this class of quantities is effected by simply adding the arcs, a property analogous to the characteristic property of logarithms.

29. If the arcs A and B be supposed equal and two imaginary factors be taken, we shall have $(\cos A \pm \sin A \sqrt{-1})^2 = \cos 2A \pm \sin 2A \sqrt{-1}$; if three factors be taken, the result will be $(\cos A \pm \sin A \sqrt{-1})^3 = \cos 3A \pm \sin 3A \sqrt{-1}$.

Whence, in general,

$$(\cos A \pm \sin A \sqrt{-1})^n = \cos nA \pm \sin nA \sqrt{-1}.$$

30. From the last equation we obtain

$$\sin nA \sqrt{-1} = (\cos A + \sin A \sqrt{-1})^n - \cos nA,$$

and $\sin nA \sqrt{-1} = -(\cos A - \sin A \sqrt{-1})^n + \cos nA$, dividing the sum of these by $2 \sqrt{-1}$, there results the following expression for the sine of any multiple arc; viz.

$$\sin nA = \frac{(\cos A + \sin A \sqrt{-1})^n - (\cos A - \sin A \sqrt{-1})^n}{2 \sqrt{-1}}.$$

An analogous process will give

$$\cos nA = \frac{(\cos A + \sin A \sqrt{-1})^n + (\cos A - \sin A \sqrt{-1})^n}{2}.$$

31. These quantities may readily be expanded into series, by means of the binomial theorem; in which case all the terms affected with imaginaries will be annihilated, and there will be obtained the two following general series, one for the *sines* the other for the *cosines* of multiple arcs.

$$\sin n\text{A} = n\cos^{n-1}\text{A}\sin\text{A} - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\text{A}\sin^3\text{A} + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\cos^{n-5}\text{A}\sin^5\text{A} - \&\text{c.} \quad \ldots.(\text{W.})$$

$$\cos n\text{A} = \cos^n\text{A} - \frac{n(n-1)}{1.2}\cos^{n-2}\text{A}\sin^2\text{A} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\text{A}\sin^4\text{A} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6}\cos^{n-6}\text{A}\sin^6\text{A} + \&\text{c.} \quad \ldots.(\text{X.})$$

32. If the arc A be supposed indefinitely small, then will $\sin\text{A} = \text{A}$, $\cos\text{A} = 1$, and that the arc $n\text{A}$ may become an assignable quantity, n itself must be regarded as indefinitely great; in that case unity will (practically) vanish in the terms $n - 1$, $n - 2$, &c. so that $n(n - 1)$ will become $= n^2$, $n(n - 1)(n - 2) = n^3$, &c. Hence, if we put $n\text{A} = \text{C}$, we shall have (since $\sin\text{A} = \text{A}$, and $\text{A} = \frac{\text{C}}{n}$), $\sin^2\text{A} = \frac{\text{C}^2}{n^2}$, $\sin^3\text{A} = \frac{\text{C}^3}{n^3}$, &c. while $\cos\text{A}$, $\cos^2\text{A}$, $\cos^3\text{A}$, will each $= 1$. The two last formulæ will, therefore, be reduced to the following, by means of which it will be easy to express the *sine* and *cosine* of any arc, in parts of that arc, or in decimals of the radius, that is, to calculate the *natural* sine and cosine of such arc.

$$\left.\begin{aligned}\sin\text{C} &= \text{C} - \frac{\text{C}^3}{1.2.3} + \frac{\text{C}^5}{1.2.3.4.5} - \frac{\text{C}^7}{1.2.3.4.5.6.7} + \&\text{c.}\\ \cos\text{C} &= 1 - \frac{\text{C}^2}{1.2} + \frac{\text{C}^4}{1.2.3.4} - \frac{\text{C}^6}{1.2.3.4.5.6} + \&\text{c.}\end{aligned}\right\}(\text{Y.})$$

These series will converge rapidly when the arc is small.[*]

33. From expressions, such as the preceding, for the sines, tangents, &c. of the sums, differences, and products of *two* arcs, or angles, it would be easy to pass to those for three, four, or more arcs. But as the properties which might thus be developed, however curious and elegant, are comparatively of little utility; we shall not present them here, but confine our investigation, either to the angles of a plane triangle, or to those which have an obvious relation to them.

A, B, and C, being the three angles of a plane triangle, since $C = 180° - (A + B)$,

$$\tan C = -\tan(A + B) = -\frac{\tan A + \tan B}{1 - \tan A \tan B},$$

whence, taking away the denominator,

$$\tan C - \tan A \tan B \tan C = -\tan A - \tan B,$$

and, by transposition,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \ldots. (4.)$$

Dividing this equation by the whole of the first member, and substituting for the products of the tangents divided by their sum, their corresponding values in cotangents (from equa. s), there will result,

$$1 = \cot A \cot B + \cot B \cot C + \cot A \cot C \ldots. (5.)$$

If the sum of A, B, and C, instead of being 180°, were 360°, the same formulæ would result, as is evident from the consideration, that in that case also we should have $\tan C = -\tan(A + B)$.[†] It may, farther, be readily seen that the same expression is applicable, so long as $A + B + C = n \cdot 180°$ or $2n \cdot 90°$.

---

[*] This branch of the subject is pursued, with great elegance, to a considerable extent, by Euler, in his valuable work, *Analysis Infinitorum.*

[†] The property restricted to the case when $A + B + C = 360°$, was announced and demonstrated by Dr. Maskelyne in the Philosophical Transactions for 1808: in the case of the triangle it had been previously demonstrated by Cagnoli.

34. Suppose $A + B + C = 90°$, then $\tan C = \cot (A + B) = \frac{1}{\tan (A + B)}$,

$$\text{or } \tan C = \frac{1 - \tan A \tan B}{\tan A + \tan B} \text{ (by equa. o.)}:$$

or, $\tan A \tan C + \tan B \tan C = 1 - \tan A \tan B$ (by mul.)
or, $1 = \tan A \tan B + \tan A \tan C + \tan B \tan C$ (by trans.)
and $\cot A \cot B \cot C = \cot A + \cot B + \cot C \ldots.$ (6.)
And this formula will obtain, in like manner, so long as $A + B + C = (2n + 1)\, 90°$.

35. Substituting for tan, its value $\frac{\sin}{\cos}$, in equa. (4) preceding, it becomes

$$\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C},$$

whence, taking away the denominators, there results,

$$\left.\begin{array}{r}\sin A \cos B \cos C + \sin B \cos A \cos C \\ + \sin C \cos A \cos B = \sin A \sin B \sin C\end{array}\right\} \ldots. (7.)$$

36. By adding $3 \sin A \sin B \sin C$ to both sides of equa. (7) we have

$$\begin{aligned}
4 \sin A \sin B \sin C &= \sin A \cos B \cos C + \sin A \sin B \sin C \\
&\quad + \sin B \cos A \cos C + \sin B \sin A \sin C \\
&\quad + \sin C \cos A \cos B + \sin C \sin A \sin B \\
&= \sin A \cos (C - B) + \sin B \cos (C - A) + \sin C \cos (B - A) \\
&= \tfrac{1}{2} \sin (A + C - B) + \tfrac{1}{2} \sin (A - C + B) + \tfrac{1}{2} \sin (B + C - A) \\
&\quad + \tfrac{1}{2} \sin (B - C + A) + \tfrac{1}{2} \sin (C + B - A) + \tfrac{1}{2} \sin (C - B + A) \\
&= \sin (A + C - B) + \sin (A + B - C) + \sin (B + C - A) \\
&= \sin 2B + \sin 2C + \sin 2A \ldots. (8.)
\end{aligned}$$

Let $A' = A + 90°$, $B' = B + 90°$, $C' = C + 90°$, then we shall have

$$4 \cos A' \cos B' \cos C' = \cos 2A + \cos 2B + \cos 2C \ldots (9.)$$

The formulæ 7, 8, 9, are due to M. Mello: the last applies to the case where $A' + B' + C' = (2n + 1)\, 90°$.

37. We may now add all that is requisite in the solution of rectilineal triangles: and shall first take the case where, instead of having two sides a, b, and their in-

cluded angle C given, the *logarithms* of those sides are given; as frequently happens in geodesic operations, and in astronomical tables for the distances of the planets from the sun. Here if a and b are regarded as the sides of a right angled triangle, in which α denotes the angle opposite to the side a, we shall have $\tan \alpha = \frac{a}{b}$. But, since a is supposed greater than b, this angle will be greater than half a right angle, or will be measured by an arc greater than 45°.

Hence, because $\tan(\alpha - 45^\circ) = \frac{\tan \alpha - \tan 45^\circ}{1 + \tan \alpha \tan 45^\circ}$, and because $\tan 45^\circ = \text{R} = 1$, we have

$$\tan(\alpha - 45^\circ) = \left(\frac{a}{b} - 1\right) \div 1 + \frac{a}{b} = \frac{a-b}{a+b}.$$

Consequently, from equa. (3)

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\text{A} - \text{B})}{\tan \frac{1}{2}(\text{A} + \text{B})} = \frac{\tan \frac{1}{2}(\text{A} - \text{B})}{\cot \frac{1}{2}\text{C}} : \text{ whence}$$

$$\tan \tfrac{1}{2}(\text{A} - \text{B}) = \cot \tfrac{1}{2}\text{C} \tan(\alpha - 45^\circ) \ldots (10.)$$

Hence results the following practical rule:—

Subtract the less from the greater of the given logs, the remainder will be the log. tangent of an angle: take 45° from that angle, and to the log. tangent of the remainder add the log. cotangent of half the given angle, the sum will be the log. tangent of half the *difference* of the other two angles of the plane triangle.

38. Let us next return to the equation (1) in art. 11 of this chapter; namely,

$$\left.\begin{aligned} a &= b \cos \text{C} + c \cos \text{B} \\ b &= a \cos \text{C} + c \cos \text{A} \\ c &= a \cos \text{B} + b \cos \text{A} \end{aligned}\right\}$$

The first of these equations being multiplied by a, the second by b, the third by c, and each of the equations thus obtained, being taken from the sum of the other two, there will arise

$$\left.\begin{array}{ll} b^2+c^2-a^2=2bc\cos A, \text{ whence } & \cos A=\dfrac{b^2+c^2-a^2}{2bc} \\ a^2+c^2-b^2=2ac\cos B, & \cos B=\dfrac{a^2+c^2-b^2}{2ac} \\ a^2+b^2-c^2=2ab\cos C, & \cos C=\dfrac{a^2+b^2-c^2}{2ab} \end{array}\right\}(11.)$$

These equations are, obviously, equivalent to prop. 17, chap. ii. They may be expressed in a form rather better suited to computation, thus:

$$\cos A=\frac{(b+c+a)(b+c-a)}{2bc}-1 \ldots\ldots (12.)$$

Where, since the expressions are perfectly symmetrical, those for cos B, cos C, need not be put down.

39. Substituting in equa. (12), for cos A its equivalent $2\cos^2\frac{1}{2}A-1$, we have

$$\cos\tfrac{1}{2}A=\sqrt{\frac{\frac{1}{2}[a+b+c]\,.\,[\frac{1}{2}(a+b+c)-a]}{bc}} \ldots\ldots (13.)$$

If, in equa. (11) we substitute $1-2\sin^2\frac{1}{2}A$ for cos A, we shall have

$$2\sin^2\tfrac{1}{2}A=1+\frac{a^2-b^2-c^2}{2bc}=\frac{(a+b-c)(a+c-b)}{2bc}.$$ Therefore

$$\sin\tfrac{1}{2}A=\sqrt{\frac{[\frac{1}{2}(a+b+c)-c]\,.\,[\frac{1}{2}(a+b+c)-b]}{bc}} \ldots (14.)$$

If, again, equa. (14) be divided by equa. (13), there will result,

$$\tan\tfrac{1}{2}A=\sqrt{\frac{[\frac{1}{2}(a+b+c)-b]\,.\,[\frac{1}{2}(a+b+c)-c]}{[\frac{1}{2}(a+b+c)-a]\,.\,[\frac{1}{2}(a+b+c)]}} \ldots (15.)$$

40. Of the rules for the determination of the angles of a plane triangle when the three sides are given, comprehended in the formulæ, marked 11, 12, 13, 14, 15, the last three are the best in practice, except the sides are integers and lie within the compass of a table of squares. When the angle sought is small, it is usually better to employ the method of equa. (14) than that of equa. (13). The method of equa. (15) is tolerably

commodious, and very correct, except when A is either very small or near 180°.

In some cases where great accuracy may be required, the student may wish to obviate the uncertainties that would arise from the use of some of these rules. For this purpose Dr. Maskelyne has given, in the introduction to Taylor's Logarithms, the following rules in reference to the sines and tangents of very small arcs.

1. *To find the sine.* To the logarithm of the arc reduced into seconds, with the decimal annexed, add the constant quantity 4·6855749, and from the sum subtract one third of the arithmetical complement of the log. cosine, the remainder will be the log. sine of the given arc.

2. *To find the tangent.* To the log. arc and above constant quantity, add two-thirds of the arithmetical complement of the log. cosine, the sum is the log. tangent of the given arc.

3. *To find the arc from the sine.* To the given log. sine of a small arc add 5·3144251, and $\frac{1}{3}$ of the arith. comp. of log. cosine: substract 10 from the index of the sum, the remainder will be the logarithm of the number of *seconds* and decimals in the given arc.

4. *To find the arc from the tangent.* To the log. tangent add 5·3144251, and from the sum subtract $\frac{2}{3}$ of the arith. comp. of log. cosine; take 10 from the index, and there will remain the logarithm of the number of seconds and decimals in the given arc.

41. Having at the end of the 2d chapter adverted briefly to the method of constructing a table of natural sines and tangents, the subject need not be resumed here. We shall merely subjoin the values of a few of the sines, cosines, &c in surd expressions, and a few formulæ of verification, such as may readily be reduced to identical equations.

Now, to radius 1, we have

$$\begin{aligned}
\sin 15^\circ &= \cos 75^\circ = \tfrac{1}{2}\sqrt{2-\surd 3} = \tfrac{1}{4}(\surd 6 - \surd 2)\\
\sin 18^\circ &= \cos 72^\circ = \tfrac{1}{4}(-1+\surd 5)\\
\sin 22\tfrac{1}{2}^\circ &= \cos 67\tfrac{1}{2}^\circ = \tfrac{1}{2}\sqrt{2-\surd 2}\\
\sin 30^\circ &= \cos 60^\circ = \tfrac{1}{2}\\
\sin 36^\circ &= \cos 54^\circ = \tfrac{1}{4}\sqrt{10-2\surd 5}\\
\sin 45^\circ &= \cos 45^\circ = \tfrac{1}{2}\surd 2\\
\sin 54^\circ &= \cos 36^\circ = \tfrac{1}{4}(1+\surd 5)\\
\sin 60^\circ &= \cos 30^\circ = \tfrac{1}{2}\surd 3\\
\sin 67\tfrac{1}{2}^\circ &= \cos 22\tfrac{1}{2}^\circ = \tfrac{1}{2}\sqrt{2+\surd 2}\\
\sin 72^\circ &= \cos 18^\circ = \tfrac{1}{4}\sqrt{10+2\surd 5}\\
\sin 75^\circ &= \cos 15^\circ = \tfrac{1}{2}\sqrt{2+\surd 3} = \tfrac{1}{4}(\surd 6 + \surd 2).
\end{aligned}$$

From these, by means of the formulæ (G), (I), (N), &c. the sines of any other arcs proposed may be computed: from these, also, may be deduced the tangents and secants.

For, since $\tan = \frac{\sin}{\cos}$, we have, for example,

$$\begin{aligned}
\tan 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\surd 6 - \surd 2}{\surd 6 + \surd 2} = 2 - \surd 3.\\
\tan 22\tfrac{1}{2}^\circ &= \frac{\sin 22\frac{1}{2}^\circ}{\cos 22\frac{1}{2}^\circ} = \frac{\surd(2-\surd 2)}{\surd(2+\surd 2)} = \surd 2 - 1.\\
\tan 30^\circ &= \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\frac{1}{2}}{\frac{1}{2}\surd 3} = \tfrac{1}{3}\surd 3.\\
\tan 67\tfrac{1}{2}^\circ &= \frac{\sin 67\frac{1}{2}^\circ}{\cos 67\frac{1}{2}^\circ} = \frac{\surd(2+\surd 2)}{\surd(2-\surd 2)} = \surd 2 + 1.\\
\tan 75^\circ &= \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\surd 6 + \surd 2}{\surd 6 - \surd 2} = 2 + \surd 3.
\end{aligned}$$

The secants may be computed from either the tangents or from the sines. Thus, from the tangents, since $\sec = \surd(\tan^2 + \mathrm{rad}^2)$, we have

$$\begin{aligned}
\sec 15^\circ &= \sqrt{(2-\surd 3)^2+1^2} = \sqrt{8-4\surd 3} = \surd 6 - \surd 2.\\
\sec 75^\circ &= \sqrt{(2+\surd 3)^2+1^2} = \sqrt{8+4\surd 3} = \surd 6 + \surd 2.\\
\sec 22\tfrac{1}{2}^\circ &= \sqrt{(\surd 2-1)^2+1^2} = \sqrt{4-2\surd 2}.\\
\sec 67\tfrac{1}{2}^\circ &= \sqrt{(\surd 2+1)^2+1^2} = \sqrt{4+2\surd 2}.
\end{aligned}$$

Or, since $\sec = \frac{1}{\cos}$, (chap. i. 19 H), we have

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{1}{\frac{1}{2}\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{\frac{1}{2}} = 2.$$

These may obviously be extended to other arcs, by means of formulæ (P), (Q), (R), of this chapter, and prop. 3 of chap. ii.

42. Operations of this kind ought not to be carried far without being subjected to checks and proofs. For this purpose, after the sines and cosines are found, the tangents, secants, &c. are easily verified by their mutual relations. The sines and cosines themselves, are examined by means of some of the following "formulæ of verification."

$$\sin(18^\circ + A) + \sin(18^\circ - A) = 2 \sin 18^\circ \cos A$$

$$\sin(54^\circ + A) + \sin(54^\circ - A) = 2 \sin 54^\circ \cos A$$

$$\sin(54^\circ + A) + \sin(54^\circ - A) - (18^\circ + A) - \sin(18^\circ - A) = \sin(90^\circ - A)$$

$$\sin A + \sin(36^\circ - A) + \sin(72^\circ + A) = \sin(36^\circ + A) + \sin(72^\circ - A).$$

Here A may have any assigned value less than the least specified arc. Thus, if A were 9°, we should have

$$\sin 9^\circ + \sin 27^\circ + \sin 81^\circ = \sin 45^\circ + \sin 63^\circ,$$

and if from any table the sines thus taken make equal sums, it is highly probable they are all correct. Taking these from Dr. Hutton's tables, we have

sin 9° =	·1564345		
sin 27° =	·4539905	sin 45° =	·7071068
sin 81° =	·9876883	sin 63° =	·8910065
	1·5981133	....equal sums....	1·5981133

43. Of the formulæ investigated in this chapter, those which have *letters* of reference (A), (B), (C), &c. relate to the sums, differences, multiples, &c. of sines, tan-

gents, &c. while those which have *figures* of reference, (1), (2), (3), &c. will be employed in the solutions of plane triangles. Other kinds will find their application in subsequent parts of this introduction; and the student will do well, after he has gone through their investigation, to arrange them in separate tables for use. We shall terminate the present chapter by subjoining three examples.

Example I.

Given the three sides of a plane triangle 40, 34, and 25, respectively, to find the largest angle, by formula (15).

Here $\frac{1}{2}(a+b+c)-c=15{\cdot}5$ .. log = 1·1903317
$\frac{1}{2}(a+b-c)-b=24{\cdot}5$ .. log = 1·3891661

Their prod. shown by the *sum* of the logs .. 2·5794978

Also, $\frac{1}{2}(a+b-c)-a=\ \ 9{\cdot}5$ .. log = 0·9777236
$\frac{1}{2}(a+b-c)\qquad\ =49{\cdot}5$ .. log = 1·6946052

Their prod. shown by the *sum* of the logs .. 2·6723288

The latter *sum* taken from the former borrowing 20 for radius squared, gives, } 19·9071690.
The latter log. ÷ by 2 for the $\surd$, gives 9·9535845.
This, by the tables, is nearly tan 41° 56½′, and by the formula it is tan ½A; therefore ½A = 41° 56½′, and the required angle = 83° 53′ nearly.

Note. In chap. iii. case 3, the same result is obtained by a different process.

Example II.

What are the angles of that plane triangle whose natural tangents are *integers?*

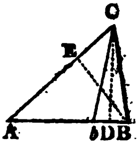
It is evident from equa. (4) that the *sum* of the three tangents must be equal to their continual *product.* Now, the only three *integers* which possess this property, are 1, 2, and 3; which are, therefore, the tangents required. The angles of which they are the tangents, are 45°, 63° 26′ 6″, and 71° 33′ 54″; whose sum makes 180°, as it ought to do.

Example III.

There is a plane triangle whose sides are three consecutive terms in the natural series of integer numbers, and whose largest angle is just double the smallest. Required its sides and angles.

That the student may compare the two methods, we shall present both a geometrical and an algebraical solution to this problem.

1st. *Geometrically.*



Analysis. Suppose the triangle ABC to be that whose sides CB, BA, AC, are respectively as three consecutive terms in the increasing series of integer numbers, and the greater angle ABC equal to twice the less angle BAC. From C upon AB let fall the perpendicular CD, make Db = DB, and join BC. Then, because the angle CbB (= ABC) is equal to 2CAB, the points A and C are in the circumference of a circle whose radius is bA, or bC, and centre b. But AB : AC + CB (= 2AB) :: AC − CB (= 2) : AD − DB = Ab $\left(= \frac{4\text{AB}}{\text{AB}} = 4\right)$ = CB; hence CB is given; and because AB = CB + 1 = AC − 1 (by hypothesis), the three sides are given, to construct the triangle.

Construction. Let the right line AB be made equal to 5, on any scale of equal parts; from centres A and B with radii 6 and 4 respectively, describe arcs to intersect each other in C; draw AC, BC, and ABC is the required triangle.

Demonstration. The sides CB, BA, AC, are three consecutive terms in the series of natural numbers (by const.) From C upon AB demit the perp. CD, set off Db = DB, and draw bC. Then, AB (= 5) : AC + CB (= 6 + 4) :: AC − CB (= 6 − 4) : AD − DB = Ab = 4. Therefore Ab = BC = bC: and hence the points A and C are in the circumference of the circle whose centre is b and radius bA, or bC; and consequently (Euc. iii. 20) angle CbB = CBA = 2 angle CAB.

2dly. *Algebraically.*

Denote BC by $x - 1$, BA by x, and AC by $x + 1$; then, (BE being perpendicular to AC the longest side) we have AC : AB + BC :: AB − BC : AE − EC, that is, $x + 1 : 2x - 1 :: 1 : \frac{2x-1}{x+1}$. Half this added to half AC, that is, $\frac{2x-1}{2x+2} + \frac{x+1}{2} = \frac{x^2+4x}{2x+2} =$ AE. But AB : rad :: AE : cos A, or, $x : 1 :: \frac{x^2+4x}{2x+2} : \frac{x+4}{2x+2} = \cos A$. Whence $\sin A = \sqrt{(1 - \cos^2 A)} = \frac{\sqrt{(3x^2-12)}}{2x+2}$.

Again, BC : sin A :: AC : $\sin B = \frac{x+1}{x-1} \cdot \frac{\sqrt{(3x^2-12)}}{2x+2}$; and (by the quest.) this angle = 2A. But, equa. (1), $\sin 2A = 2 \cos A \sin A = \frac{x+4}{x+1} \cdot \frac{\sqrt{(3x^2-12)}}{2x+2} = \sin B$.

By equating these two values of sin B, we have $\frac{x+1}{x-1} = \frac{x+4}{x+1}$; whence $x^2 + 2x + 1 = x^2 + 3x - 4$, and $x = 5$. The sides, therefore, of the triangle are 4, 5, and 6. The cosines of the angles are, $\cos A = \frac{3}{4}$, $\cos C = \frac{9}{16}$, $\cos B = \frac{1}{8}$. The sines are, $\sin A = \frac{1}{4}\sqrt{7}$, $\sin C = \frac{5}{16}\sqrt{7}$, $\sin B = \frac{3}{8}\sqrt{7}$.

And the angles are, $A = 41^\circ\ 24'\ 34''\ 34'''$
$C = 55^\circ\ 46'\ 16''\ 18'''$
$B = 82^\circ\ 49'\ 9''\ 8'''$

CHAPTER V.

Application of Plane Trigonometry to the determination of Heights and Distances.

1. ONE of the most familiar, and at the same time, useful, applications of plane trigonometry, is to the determination of the altitudes and distances of remote objects, the former usually designated by the term *altimetry*, the latter by that of *longimetry*. It is not intended to treat them separately here, nor to treat them jointly much in detail; but simply to present and solve a selection of such problems as are most likely to occur in practice, and as are best calculated to suggest the modes of procedure in other cases.

2. The instruments employed to measure angles are quadrants, sextants, and other circular instruments. For military men, perhaps, the best instrument is a pocket sextant, which, if accurately constructed by a skilful artist, will enable a careful observer to take angles to within a minute of the truth. But for general purposes the most proper instrument for the measuring of horizontal and vertical angles, is a theodolite furnished with a compass and level, one or two telescopes, and a vertical arc. Such an instrument, when each circle is 6 or 7 inches diameter, and has a nonius adapted to it, will enable the observer to ascertain angles to half a minute.

3. The space would be improperly occupied in giving particular descriptions of these instruments, and the manner of adjusting them for use. These matters will be much better learnt from an examination of the instruments, and a few explanatory remarks from a judicious tutor. Nor shall we here embarrass the computation and diagrams, by showing how to allow for the *height* of the instrument, for that is a matter which requires only a single hint from the person who teaches the use of the instrument employed. So again, in reference to chains, tapes, and other contrivances for measuring lines, descriptions are suppressed. It ought, however, to be observed, that whenever a base, or distance between two stations, is measured on sloping ground, it must be reduced to the corresponding horizontal line, before it is employed in the general computation, if horizontal angles are taken at its extremities with a theodolite. This is in all cases easily effected: for, if the sloping or hypothenusal line be regarded as radius, the corresponding horizontal line will be the cosine of the inclination of the plane on which the sloping line was measured. It is, therefore, simply requisite to multiply the *natural cosine* of the angle of inclination, by the length of the line measured, to obtain the true horizontal line.

Suppose that on a plane inclined to the horizon in an angle of 4½ degrees, a distance of 400 yards were measured, what would be the corresponding horizontal distance?

Multiply nat. cos 4½° = ·9969173
By 400

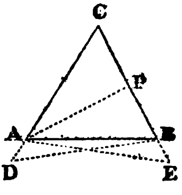
Product gives hor. dist. = 398·7669200

Here the difference is not 1¼ yard, or not a 320th part of the measured line. As this is not a greater deviation from accuracy than will occur in the usual processes for measuring distances with a chain or a tape, the reduction from the sloping to the horizontal line, may, in common cases, be disregarded, except the angle of inclination exceed $4\frac{1}{2}°$ or $5°$.

4. There are several simple methods of approximating to the heights of both accessible and inaccessible objects, by means of shadows, mirrors, unequal vertical staffs, &c. But as these depend solely upon the principle of similar triangles, and do not require the theorems and formulæ of trigonometry, properly so called, they are not described here. The student may, without being detained by farther observations, proceed to the solution of the following problems.

EXAMPLE I.

In order to ascertain my distance from a tower, which was inaccessible by reason of an intervening river, I measured on a horizontal plane a base line of 600 yards, and at each end took the angle included between the other end and the tower, finding them to be 57° 35′ and 64° 51′ respectively. Required the distance of the tower from each end of the measured base.



In the annexed figure are given

$AB = 600$
$CAB = 57° 35'$
$CBA = 64° 51'$
conseq. $C = 180° - (A + B) = 57° 34'$.

Logs
Hence, sin C .... 57° 34′ .... 9·9263507
To AB ... 600 .... 2·7781513
So is sin A 57° 35′ .... 9·9264310
To BC ... 600·11 .... 2·7782316
and, So is sin B 64° 51′ .... 9·9567437
To AC ... 643·49 .... 2·8085443.

Remark. These distances might have been determined without the aid of an instrument to measure angles. Thus, suppose in the continuations of the respective lines CA, CB, two distances, AD, BE were measured, each = 120; and suppose, on measuring the distances BD, AE, the former is found to be 660, the latter 672 yards; from these measures the required angles may be determined.

For, in the triangle ABE thus formed, we have (Euc. ii. 12) $AE^2 = AB^2 + BE^2 + 2EB . BP$. Whence, by transposition and division, $BP = \frac{AE^2 - AB^2 - BE^2}{2EB}$. But BP is the cosine of the angle ABC, to the radius AB; so that, dividing the preceding expression by AB, we shall have the cosine of that angle to radius 1. A like process will give the cosine of CAB.

$$\text{Thus, } \cos ABC = \frac{AE^2 - AB^2 - BE^2}{2EB . BA} = \frac{672^2 - 600^2 - 120^2}{240 . 600}$$
$$= \cdot 425 = \cos 64^\circ 51'$$
$$\text{and, } \cos BAC = \frac{BD^2 - AB^2 - AD^2}{2AD . AB} = \frac{660^2 - 600^2 - 120^2}{240 . 600}$$
$$= \cdot 536 = \cos 57^\circ 35'.$$

Hence, the angles ABC, and BAC, being determined, the distances are found as before.

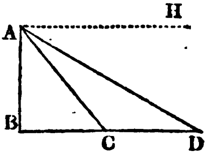
EXAMPLE II.

In order to find the distance between two trees A and B, which could not be directly measured because of a pool which occupied much of the intermediate space, I measured the distance of each of them from a third object C, viz. AC = 588, BC = 672, and then at the point C took the angle ACB between the two trees = 55° 40′. Required their distance.

This is an example to case 2 of plane triangles, in which two sides and the included angle are given. The work, therefore, is left to exercise the student: the answer is 593·8

EXAMPLE III.

Wanting to know the distance between two inaccessible objects, which lay in a direct line from the bottom of a tower on whose top I stood, I took the angles of *depression* of the two objects, viz. of the most remote 25½°, of the nearest 57°. What is the distance between them, the height of the tower being 120 feet?



The figure being constructed, as in the margin, AB = 120 feet, the altitude of the tower, and AH the horizontal line drawn through its top; there are given,

HAD = 25° 30′, hence BAD = BAH − HAD = 64° 30′
HAC = 57° 0′, hence BAC = BAH − HAC = 33° 0′.

Hence the following calculation.

In right angled △ ABC.		In right angled △ ABD.	
As rad	10·0000000	As rad	10·0000000
To AB .. 120 ..	2·0791812	To AB .. 120 ..	2·0791812
So is tan BAC 33°	9·8125174	So is tan BAD 64½°	10·3215039
To BC .. 77·929	1·8916986	To BD .. 251·585	2·4006851

Conseq. BD − BC = 173·656 feet, the distance required.

Such is the way in which this problem is usually solved: it may, however, be done more easily and concisely, by means of the *natural* tangents. For, if AB be regarded as radius, BD and BC will be the tangents of the respective angles BAD, BAC, and CD the difference of those tangents. It is, therefore, equal to the product of the difference of the natural tangents of those angles into the height AB.

Thus, nat. tan 64½° =	2·0965436
nat. tan 33° =	0·6494076
difference	1·4471360
multiplied by height .	120
gives distance CD	173·6563200

EXAMPLE IV.

From the top of a hill I observed two mile-stones on a horizontal road, which ran straight from its bottom, and took their respective angles of depression below the horizontal plane passing through the place of my eye; that of the nearer mile-stone was 14° 3′, that of the farther was 3° 56′. Required the height of the hill.

The figure being drawn, it will be found analogous to that in exam. 3, to which, therefore, we shall refer. There are given CD = 1760 yards, the distance between the two mile-stones; ADB = HAD = 3° 56′; ACB = HAC = 14° 3′. This admits of three distinct modes of determination, as below.

1st *Method.*

The angle ACB is equal to the sum of the angles CAD and CDA (Euc. i. 32): therefore CAD = ACB − ADB = 10° 7′. Then, in the triangle ACD, it will be, as sin CAD : CD :: sin CDA : CA. And, in triangle ACB, it will be, as rad. or sin B : AC :: sin C : AB = 166·85; and so, if it be required, is sin CAB : CB = 666·75.

According to this method the logarithmic process will require *eight* lines.

2d *Method.*

Since CAD is the difference of ACB and ADC, we have, $\sin (C - D) : CD :: \sin D : AC = \frac{CD \,.\, \sin D}{\sin (C-D)}$. Also, rad : AC $\left(= \frac{CD \sin D}{\sin (C-D)}\right) :: \sin C : AB = \frac{CD \sin C \sin D}{\sin (C-D)}$. But (ch. i. 19 H) $\operatorname{cosec} (C - D) = \frac{1}{\sin (C-D)}$. Hence $AB = CD \sin C \sin D \operatorname{cosec} (C - D)$.

Or, making the terms homogeneous (ch. iv. 17),

$$AB \,.\, \text{rad}^3 = CD \sin C \sin D \operatorname{cosec} (C - D).$$

The logarithmic formula is, therefore, this:

$\log AB = \log CD + \log\sin C + \log\sin D + \log\mathrm{cosec}\,(C - D)$
$- 30$ [in the index of the log].

Thus, log CD	1760	3·2455127
log sin C	14° 3′ ..	9·3851924
log sin D	3° 56′ ..	8·8362969
log cosec (C—D)	10° 7′ ..	10·7553442
The sum — 30, is log AB	166·857 feet	2·2223462

This method is, obviously, applicable to all similar examples.

3d *Method.*

If AB were radius, CD would evidently be the difference of the tangents of BAD and BAC, or, of the cotangents of BDA and BCA. Hence $AB = \frac{CD}{\cot D - \cot C}$.

Thus, nat cot D = 14·543833
nat cot C = 3·995922

```
10·547911)1760·000(166·857ft. as bef.
   ......  1054791
           -------
            705209
            632875
            ------
             72334
             63287
             -----
              9047
              8438
              ----
               609
               527
               ---
                82
                74
                --
```

From the comparison of these three methods, it will appear that the second ought to have the preference to the first: and that, considering the time employed in looking out the logarithms, the third method is preferable to the second.

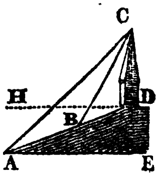
EXAMPLE V.

Wanting to know the height of a church steeple, to the bottom of which I could not measure on account of a high wall between me and the church, I fixed upon two stations at the distance of 93 feet from each other, on a horizontal line from the bottom of the steeple, and at each of them took the angle of elevation of the top of the steeple, that is, at the nearest station 55° 54′, at the other 33° 20′. Required the height of the steeple.

This is similar to exam. 4, and being worked by the 2d method, the height of the steeple is found to be 110·27 feet.

EXAMPLE VI.

Wishing to know the height of an obelisk standing at the top of a regularly sloping hill, I first measured from its bottom a distance of 36 feet, and there found the angle formed by the inclined plane and a line from the centre of the instrument to the top of the obelisk 41°; but after measuring on downward in the same sloping direction 54 feet farther, I found the angle formed in like manner to be only 23° 45′. What was the height of the obelisk, and what the angle made by the sloping ground with the horizon?



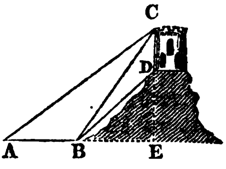
The figure being constructed, as in the margin, there are given in the triangle ACB, all the angles and the side AB, to find BC. It will be obtained by this proportion, as sin C (= 17° 15′ = B − A) : AB (= 54) :: sin A (= 23° 45′) : BC = 73·3392. Then, in the triangle DBC are known BC as above, BD = 36, CBD = 41; to find the other angles, and the side CD. Thus, first, as CB + BD : CB − BD :: tan $\frac{1}{2}$ (D + C) = $\frac{1}{2}$ (139°) : tan $\frac{1}{2}$ (D − C) = 42° 24$\frac{1}{2}$′. Hence 69° 30′ + 42° 24$\frac{1}{2}$′ = 112° 54$\frac{1}{2}$° = CDB, and 69° 30′ − 42° 24$\frac{1}{2}$′ = 26° 5$\frac{1}{2}$′ = BCD. Then, sin BCD : BD :: sin CBD : CD = 51·86, height of the obelisk.

The angle of inclination DAE = HDA = CDB − 90° = 22° 54$\frac{1}{2}$′.

Remark. If the line BD cannot be measured, then the angle DAE of the sloping ground must be taken, as well as the angles CAB, and CBD. In that case DAE + 90° will be equal to CDB: so that after CB is found from the triangle ACB, CD may be found in the triangle CBD, by means of the relation between sides and the sines of their opposite angles.

EXAMPLE VII.

Being on a horizontal plane, and wanting to ascertain the height of a tower standing on the top of an inaccessible hill, I took the angle of elevation of the top of the hill 40°, and of the top of the tower 51°, then measuring in a direct line 180 feet farther from the hill, I took in the same vertical plane the angle of elevation of the top of the tower 33° 45′. Required from hence the height of the tower.



The figure being constructed, as in the margin, there are given, AB = 180, CAB = 33° 45′ ACB = CBE − CAE = 17° 15′ CBD = 11°, BDC = 180° − (90° − DBE) = 130°. And CD may be found by two proportions, viz. 1st. As sin ACB : AB :: sin CAB : CB, and 2dly, as sin D : CB :: sin CBD : CD. This process would require eight lines. But the operation may be shortened; for, by the principles of method 2, exam. 4, we shall have

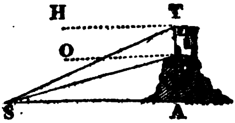
$$\text{CD rad}^4 = \text{AB sin A sin CBD cosec ACB sec DBE}.$$

In logarithms thus: log AB	180	2·2552725
sin A	33° 45′	9·7447390
sin B	11° 0′	9·2805988
cosec AGF	17° 15′	10·5279144
sec DBE	40° 0′	10·1157460
The sum − 40, is log CD	83·9983 feet	1·9242707

EXAMPLE VIII.

At the top of a castle which stood on a hill near the sea-shore, the angle of depression of a ship at anchor was observed to be 4° 52′; at the bottom of the castle the angle of depression was 4° 2′. Required the horizontal distance of the vessel, and the height of the hill on which the castle stands above the level of the sea, the castle itself being 60 feet high.

In the annexed diagram, where HT, OB, are parallel, and AT perpendicular to the horizontal line AS, are given BT = 60 feet, HTS = 4° 52′, consequently ATS = 85° 8′, and OBS = 4° 2′, whence ABS = 85° 58′; to find AS and AB. Here method 2, exam. 4, is obviously applicable: so that we have



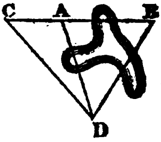
$$\text{AS} \cdot \text{rad}^3 = \text{BT} \sin \text{ATS} \sin \text{ABS} \operatorname{cosec} (\text{ABS} - \text{ATS})$$
$$\text{AB} \cdot \text{rad}^3 = \text{BT} \sin \text{ATS} \cos \text{ABS} \operatorname{cosec} (\text{ABS} - \text{ATS}).$$

The logarithmic operation will stand thus:

log TB	60	1·7781513	log TB	60	1·7781513
sin ATS	85° 8′	9·9984315	sin ATS	85° 8′	9·9984315
sin ABS	85° 58′	9·9989230	cos ABS	85° 58′	8·8471827
cosec (B − T)	50′	11·8371392	cosec (B − T)	50′	11·8373192
log AS	4100·4 ft.	3·6128250	log AB	289·12 ft.	2·4610847

EXAMPLE IX.

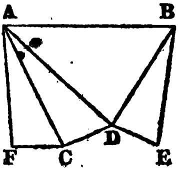
Wanting to know the distance of an object at D from two others A and B in the same horizontal plane, as well as the distance between A and B, a pole was set up at C in a right line with AB, and the angle ACD was found to be 57°. The distance CD being measured was found to be 549·36 yards; and at D the angles CDA and ADB were taken; the first = 14°, the latter = 41° 30′. Required the above specified distances.



Here, the sum of the angles C and CDA taken from 180°, leaves the angle CAD, and the sum of the angles C and CDB taken from 180°, leaves the angle CBD. Then, it will be as sin CBD : CD :: sin C : DB. As sin CAD : CD :: sin C : DA. And sin ABD : AD :: sin ADB : AB. These operations will give DB = 498·68, DA = 487·27, AB = 349·52 yards.

EXAMPLE X.

Wanting to know the horizontal distance between two inaccessible objects A and B, and not finding any station from which I could see them both, I chose two points C and D distant 200 yards, from the former of which A could be seen, from the latter B, and at each of the points C and D a flag-staff was set up. I then measured FC = 200, DE = 200, and, having set up flag-staves at F and E, took the following angles, viz. AFC = 83°, ACF = 54° 31′, ACD = 53° 30′, BDC = 156° 25′, BDE = 54° 30′, and BED = 88° 30′. Required AB.

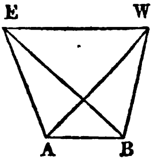


Here, in the triangle AFC, all the angles are given, and the side FC, to find AC. Then, in the triangle ACD are given AC, CD, and the contained angle, to find the other angles and the side AD. Next, in the triangle

BDE are given DE and all the angles to find DB. Lastly, in the triangle ADB are known AD, DB, the included angle BDA = BDC − ADC, to find AB = 345·5 yards.

EXAMPLE XI.

In order to determine the distance between two inaccessible objects E and W, on a horizontal plane, we measured a convenient base AB of 536 yards, and at the extremities A and B took the following angles, viz. BAW = 40° 16′, WAE = 57° 40′, ABE = 42° 22′, EBW = 71° 7′. Required the distance EW.

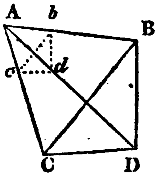


First, in the triangle ABE are given all the angles, and the side AB, to find BE. So again, in the triangle ABW, are given all the angles and AB to find BW, Lastly, in the triangle BEW are given the two sides EB, BW, and the included angle EBW, to find EW = 939·52 yards.

Remark. In like manner the distances taken two and two, between any number of remote objects posited around a convenient station line, may be ascertained.

EXAMPLE XII.

Suppose that in carrying on an extensive survey, the distance between two spires A and B has been found equal to 6594 yards, and that C and D are two eminences conveniently situated for extending the triangles, but not admitting of the determination of their distance by actual admeasurement: to ascertain it, therefore, we took at C and D the following angles, viz.



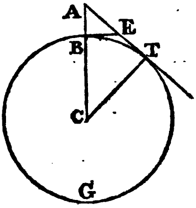
ACB = 85° 46′ ADC = 31° 48′
BCD = 23° 56′ ADB = 68° 2′

Required CD from these data.

In order to solve this problem, construct a similar quadrilateral A*cdb*, assuming *cd* equal to 1, 10, or any other convenient number: compute A*b* from the given angles, according to the method of the preceding example. Then, since the quadrilaterals A*cdb*, ACDB, are similar, it will be, as A*b* : *cd* :: AB : CD; and CD is found = 4694 yards.

EXAMPLE XIII.

If the height of the mountain called the Pike of Teneriffe be 3 miles, and the angle formed at its top between the vertical or plumb line and a right line conceived to touch the earth in the horizon, or at the farthest visible point, be 87° 46′ 33″; it is required from hence to determine the diameter of the earth, supposing it to be a perfect sphere.



Let C, in the marginal diagram, be the centre of the earth, the circle BTG a vertical section passing through the centre, AB the height of the Pike of Teneriffe, and AT the tangential line drawn to the visible horizon: let, also, BE, a tangent to the earth's surface at B, meet the other tangent AT in E. Then, in the triangle ABE, right angled at B, and having the angle BAE = 87° 46′ 33″, it is, as rad : AB :: tan A : BE, and :: sec A : AE. But it is evident, from Euc. iii. 37, that BE = ET; therefore, AE + EB = AE + ET = AT. In the triangle ATC right angled at T, we have, as rad : AT :: tan A : TC, the radius of the earth.

The operation performed as above described occupies but small compass; it may, however, be shortened by means of ch. ii. prop. 4. For, since tan A + sec A = tan (A + $\frac{1}{2}$ comp. A), we shall, by incorporating the proportions from which AE, BE, and CT are deduced, have

$$\text{CT} \cdot \text{rad}^2 = \text{AB} \tan\left(\text{A} + \tfrac{1}{2}\,\text{comp. A}\right) \tan \text{A}.$$

$$\text{Or, } \log \text{CT} = \log \text{AB} + \log \tan\left(\text{A} + \tfrac{1}{2}\,\text{comp. A}\right)$$
$$+ \log \tan \text{A} - 20, \text{ in the index.}$$

Thus, log AB ....3......... 0·4771213
log tan (A + ½ comp. A) 88° 53′ 16½″ 11·7119309
log tan A ....87° 46′ 33″ ..11·4107381

log CT ....3979·15.... 3·5997968

The double of this, or 7958·3 miles, is the diameter of the earth.

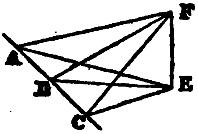
If AT be required, we have only to take radius (10) from the sum of the first two lines, the remainder 2·1890522, is the log of 154·54 the distance sought.

Note 1. The log tangents are found in this example by the method taught at p. 153 of the Introduction to Dr. Hutton's Logarithm Tables.

Note 2. This method of determining the earth's radius, though elegant in theory, is rendered useless in practice, by the extreme irregularity of the horizontal refractions.

EXAMPLE XIV.

Given the angles of elevation of any distant object, taken at three places in a horizontal right line, which does not pass through the point directly below the object; and the respective distances between the stations; to find the height of the object, and its distance from either station.



Let ABC be the horizontal plane, FE the perpendicular height of the object above that plane, A, B, C, the three places of observation, FAE, FBE, FCE, the angles of elevation, and AB, BC, the given distances. Then, since the triangles AEF, BEF, CEF, are all right angled at E, the distances AE, BE, CE, will manifestly be as the cotangents of the angles of elevation at A, B, and C.

Put AB = D, BC = d, EF = x, and then express algebraically the following theorem, demonstrated at p. 128, *Simpson's Select Exercises*: viz.

$$\text{AE}^2 . \text{BC} + \text{CE}^2 . \text{AB} = \text{BE}^2 . \text{AC} + \text{AC} . \text{AB} . \text{BC},$$

the line EB being drawn from the vertex E of the triangle ACE, to any point B in the base. The equation thence resulting is,

$$dx^2 \cot^2\text{A} + \text{D}x^2 \cot^2\text{C} = (\text{D} + d)x^2 \cot^2\text{B} + (\text{D} + d)\,\text{D}d.$$

Hence, transposing all the unknown terms to one side of the equation, dividing by the sum of the coefficients, and extracting the square root, we shall have

$$x = \sqrt{\frac{(\text{D} + d)\,\text{D}d}{d \cot^2\text{A} + \text{D} \cot^2\text{C} - (\text{D} + d) \cot^2\text{B}}}.$$

Thus EF becoming known, the distances AE, BE, CE, are found, by multiplying the cotangents of A, B, and C, respectively, by EF.

Cor. When $\text{D} = d$, or $\text{D} + d = 2\text{D} = 2d$, the expression becomes

$$x = d \div \sqrt{(\tfrac{1}{2} \cot^2\text{A} + \tfrac{1}{2} \cot^2\text{C} - \cot^2\text{B})},$$

which is pretty well suited for logarithmic computation. The rule may, in that case, be thus expressed.—

Double the log. cotangents of the angles of elevation of the extreme stations, find the natural numbers answering thereto, and take half their sum; from which subtract the natural number answering to twice the log. cotangent of the middle angle of elevation: then half the log. of this remainder subtracted from the log. of the measured distance between the 1st and 2d, or the 2d and 3d stations, will be the log. of the height of the object.*

To illustrate these general methods, two particular examples are subjoined.

EXAMPLE XV.

Let the three angles of elevation be 36° 50′, 21° 24′, and 14°, and the two equal measured distances 84 feet. Required the height of the object.

---

* For geometrical constructions of this general problem the student may consult *Hutton's Course*, vol. iii. p. 128, *Leybourn's Ladies' Diaries*, vol. i. p. 397, of the *Ladies' Diary* for 1748.

Ans. 53·964 feet.

EXAMPLE XVI.

Let the distances be AB = 100 yards, BC = 400 yards, and the angles, at A = 5° 24′, at B = 6° $27\frac{1}{2}$′, at C = 8° 36′. What is the height of the object?

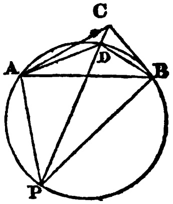
Ans. 44·46 yards.

EXAMPLE XVII.

From a convenient station P, where could be seen three objects A, B, and C, whose distances from each other were known (viz. AB = 800, AC = 600, BC = 400 yards), I took the horizontal angles APC = 33° 45′, BPC = 22° 30′. It is hence required to determine the respective distances of my station from each object.

Here it will be necessary, as preparatory to the computation, to describe the manner of

Construction. Draw the given triangle ABC from any convenient scale. From the point A draw a line AD to make with AB an angle equal to 22° 30′ and from B a line BD to make an angle DBA = 33° 45′. Let a circle be described to pass through their intersection D, and through the points A and B. Through C and D draw a right line to meet the circle again in P: so shall P be the point required. For, drawing PA, PB, the angle APD is evidently = ABD, since it stands on the same arc AD: and for a like reason BPD = BAD. So that P is the point where the angles have the assigned value.



Manner of computation. In the triangle ABC where the sides are known, find the angles. In the triangle ABD, where all the angles are known, and the side AB, find one of the other sides AD. Take BAD from BAC, the remainder, DAC is the angle included between two known sides AD, AC; from which the angles ADC and ACD may be found, by chap. iii. case 2. The angle $\text{CAP} = 180^\circ - (\text{APC} + \text{ACD})$. Also, $\text{BCP} = \text{BCA} - \text{ACD}$; and $\text{PBC} = \text{ABC} + \text{PBA} = \text{ABC} + \text{sup. ADC}$. Hence, the three required distances are found by these proportions. As $\sin \text{APC} : \text{AC} :: \sin \text{PAC} : \text{PC}$, and $:: \sin \text{PCA} : \text{PA}$; and lastly, as $\sin \text{BPC} : \text{BC} :: \sin \text{BCP} : \text{BP}$. The results of the computation are, $\text{PA} = 709{\cdot}33$, $\text{PC} = 1042{\cdot}66$, $\text{PB} = 934$ yards.

⁂ The computation of problems of this kind, however, may be a little shortened by means of the following

General Investigation.

Put $\text{AC} = a$, $\text{BC} = b$, $\text{APC} = \text{P}$, $\text{BPC} = \text{P}'$, $\text{ACB} = \text{C}$, and let there be taken for unknown quantities $\text{PAC} = x$, $\text{PBC} = y$. The triangles PAC and PBC give

$$\sin \text{APC} : \sin \text{CAP} :: \text{AC} : \text{CP}$$

$$\text{and } \sin \text{BPC} : \sin \text{CBP} :: \text{BC} : \text{CP};$$

$$\text{that is, } \sin \text{P} : \sin x :: a : \frac{a \sin x}{\sin \text{P}} = \text{CP}$$

$$\text{and } \sin \text{P}' : \sin y :: b : \frac{b \sin y}{\sin \text{P}'} = \text{CP}.$$

Hence $\dfrac{a \sin x}{\sin \text{P}} = \dfrac{b \sin y}{\sin \text{P}'}$; which reduces to

$$a \sin \text{P}' \sin x - b \sin \text{P} \sin y = 0.$$

In the quadrilateral ACBP, we have

$$\text{CBP} = 360^\circ - \text{APC} - \text{BPC} - \text{ACB} - \text{CAP}$$

$$\text{or } y = 360^\circ - \text{P} - \text{P}' - \text{C} - x.$$

Make $360^\circ - \text{P} - \text{P}' - \text{C} = \text{R}$, then we shall have $y = \text{R} - x$, and consequently,

$$a \sin \text{P}' \sin x - b \sin \text{P} (\sin \text{R} \cos x - \cos \text{R} \sin x) = 0.$$

Dividing by $\sin x$ there results,

$$a \sin \text{P}' - b \sin \text{P} \left(\sin \text{R} \frac{\cos x}{\sin x} - \cos \text{R}\right) = 0.$$

Whence we have

$$\frac{\cos x}{\sin x} = \cot x = \frac{a \sin \mathrm{P}' + b \sin \mathrm{P} \cos \mathrm{R}}{b \sin \mathrm{P} \sin \mathrm{R}}.$$

This expression being separated into two parts, we have

$$\cot x = \frac{a \sin \mathrm{P}'}{b \sin \mathrm{P} \sin \mathrm{R}} + \frac{\cos \mathrm{R}}{\sin \mathrm{R}};$$

$$\text{or, } \cot x = \frac{\cos \mathrm{R}}{\sin \mathrm{R}} \left(\frac{a \sin \mathrm{P}'}{b \sin \mathrm{P} \cos \mathrm{R}} + 1 \right);$$

$$\text{or, lastly, } \cot x = \cot \mathrm{R} \left(\frac{a \sin \mathrm{P}'}{b \sin \mathrm{P} \cos \mathrm{R}} + 1 \right);$$

Hence, x being thus determined, we get y from the equation $y = \mathrm{R} - x$, and CP from either of the expressions above given.

Note. It will be a useful exercise for the student, to work out the computation by both these methods. The comparison of the results will serve to give him confidence in the deductions from the analytical investigations.

EXAMPLE XVIII.

It is required to find the distances from Edystone light-house to Plymouth, Start Point, and the Lizard, respectively, from the following data:

The distance from	miles.
Plymouth to Lizard....	60
Lizard to Start Point ..	70
Start Point to Plymouth	20

bears from Edystone rock	
Plymouth	North.
Lizard	W.S.W.
Start Point...	E. by N.

Ans. From Edystone to Lizard ..... 53·04 miles
to Plymouth .. 14·333 ditto
to Start Point.. 17·36 ditto.

Remark. The general problem of which the last two examples are particular cases, was originally proposed by *Richard Townley*, Esq. and solved by Mr. *John Collins*, in the Philosophical Transactions, N° 69, A. D. 1671. See *New Abridgement*, vol. i. p. 563.

The six cases there considered are, 1. When the station is out of the triangle made by the objects, but in one of its sides produced. 2. When the station is in one of the sides of the triangle. 3. When the three objects lie in a right line. 4. When the station is *not* within the triangle formed by the objects. 5. When the station *is* within that triangle. 6. When, by reason of the relation of the sides and angles, the points C and D (see the preceding diagram) fall so near together as to make the continuation to P of doubtful accuracy. To these, later writers on trigonometry have added another case, viz. 7. When the point C falls between the line AB and the station P.

It will be advantageous to the student, to modify the construction and computation to suit all these cases.

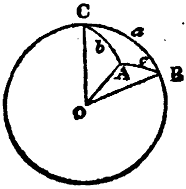
CHAPTER VI.

Spherical Trigonometry.

Section I.

Fundamental Principles, General Properties, and Formulæ.

1. THREE planes, AOC, AOB, BOC, all of which pass through the centre O of a sphere, intersect the surface of that sphere in portions of great circles which form a *spherical triangle* ABC. Thus also is constituted the spherical pyramid or tetraedron which has for its base the triangle ABC, and for its vertex the centre O of the sphere. The angle A



of the triangle, is the same as the diedral angle, between the two planes BAO, CAO; it is, also, the angle formed by the tangents to the two arcs AC, AB. The like may be said of the other *angles.* The *sides* are manifestly the measures of dependent plane angles, viz. *a* the measure of the angle COB, *b* the measure of COA, *c* the measure of AOB.

2. A *right angled* spherical triangle has one right angle; the sides about the right angle are called *legs;* the side opposite to the right angle is called the *hypothenuse.*

3. A *quadrantal* spherical triangle has one side equal to 90°, or is a quarter of a great circle.

4. An *isosceles*, or an *equilateral* spherical triangle, has respectively *two* sides or *three* sides equal.

5. When the sides of a triangle are each 90°, it is not only an *equilateral*, but a *quadrantal*, and a *right angled* triangle. All its angles as well as its sides are equal; and these sides may any of them be regarded as an *hypothenuse*, any of them as *legs.* Such is the case with the triangle that would be formed on a celestial or terrestrial globe, by the horizon, the brazen meridian, and a quadrant of altitude, fixed at the zenith, and passing through the east or west point.

6. Two arcs or angles, when compared together, are said to be *alike*, or of the *same affection*, when both are less, or both greater than 90°. They are said to be *unlike*, or of *different affections*, when one is greater and the other less than 90°.

7. Every spherical triangle has three sides and three angles; of which, if any three be given, the remaining three may be found.

8. In plane trigonometry, the knowledge of the three angles is not sufficient for ascertaining the sides (chap. i. 6): but, in spherical trigonometry, the sides may always be determined when the angles are known. In plane triangles, again, two angles always determine the third; in spherical triangles they never do. So, farther,

the surface of a plane triangle cannot be determined from its angles merely; that of a spherical triangle always can.

9. A line perpendicular to the plane of a great circle, passing through the centre of the sphere, and terminated by two points diametrically opposite, at its surface, is called the *axis* of such circle; and the extremities of the axis, or the points where it meets the surface, are the *poles* of that circle.

If we conceive any number of less circles, each parallel to the said great circle, this axis will be perpendicular to them likewise; and the poles of the great circle will be their poles.

10. Hence, each pole of a great circle is 90° distant from every point in its circumference; and all the arcs drawn from either pole of a little circle to its circumference, are equal to each other.

11. It likewise follows that all the arcs of great circles, drawn through the poles of another great circle, are perpendicular to it; for, since they are great circles by the supposition, they all pass through the centre of the sphere, and consequently through the axis of the said circle. The same thing may be affirmed in reference to small circles.

12. Hence, in order to find the *poles* of any circle, it is merely necessary to describe, upon the surface of the sphere, two great circles perpendicular to the plane of the former, the points where these circles intersect each other will be the poles required.

13. All great circles *bisect* each other. For, as they have a common centre, their common section will be a diameter; and that manifestly bisects them.

14. The small circles of the sphere do not fall under consideration in spherical trigonometry; but such only as have the same centre with the sphere itself. Hence appears the reason why spherical trigonometry is of such great use in practical astronomy, the apparent heavens being regarded as in the shape of a concave

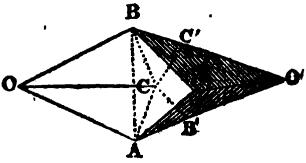
sphere having its centre either at the centre of the earth, or at the eye of the observer.

PROBLEM.

15. To investigate properties and equations from which the solution of the several cases of spherical trigonometry may be deduced.

In order to this let us recur to the spherical tetraedron OABC, where the angles A, B, C, of the spherical triangle are the diedral angles between each two of the three planes AOC, AOB, &c. and the sides a, b, c, are the measures of the plane angles COB, COA, &c. ' Here it is 1st, evident that the three sides of a spherical triangle are together less than a circle, or, $a + b + c < 360°$. For the solid angle at O is contained by three plane angles, which (Euc. xi. 21) are together less than four right angles; therefore, the sides a, b, c, which measure those plane angles are together less than a circle.

16. Let the tetraedron OABC be cut by planes perpendicular to the three edges; they will form another tetraedron O′A′B′C′; their faces will be respectively perpendicular, two and two. But, in the quadrilateral AOBA′ since the angles A and B are right angles, the plane angle O is the *supplement* of AA′B which measures the diedral angle AA′O′B. The same may be shown with respect to the other plane angles that meet at O; as well as of the plane angles at O′, in reference to the diedral angles of the tetraedron OABC. Therefore, either of these tetraedrons, has *each of its plane angles supplement to a diedral angle in the other:* it is hence called the *supplementary tetraedron.* And if they become spherical tetraedrons referred to equal spheres, or to different parts of the same sphere, their bases will be spherical triangles respectively *supplemental* to each other.



17. It is obvious from this that the problems in spherical trigonometry become susceptible of reduction to half their number; since, if there are given, for example, the three angles A, B, C, and the three sides a, b, c, are required; let the triangle which has for its sides a', b', c', the supplements of the measures of A, B, and C, have its angles A$'$, B$'$, C$'$, determined; their measures will be the supplements of the required sides a, b, and c.

18. On the surface of the sphere, the *supplemental triangle* is formed by the intersections of three great circles described from the angles of the primitive triangle as poles. Besides the supplemental triangle, three others are formed in each hemisphere by the mutual intersections of these three great circles; but it is the *central* triangle (of each hemisphere) that is supplementary.

19. Every angle between two planes being less than two right angles, it follows, that *the sum of the angles of a spherical triangle is less than 3 times 2, or than 6 right angles.* At the same time, it is *greater than 2 right angles:* for the sum $a' + b' + c'$ of the sides of the supplemental triangle is less than 360° (art. 15 above): taking the supplements, we have

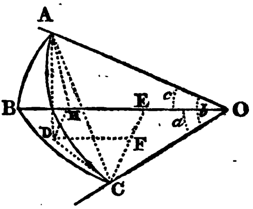
$$3 \times 180^\circ - (a' + b' + c') > 180^\circ, \text{ or } A + B + C > 180^\circ.$$

20. To deduce the fundamental theorems, we may proceed thus. From any point A of the edge AO of the tetraedron, let fall on the plane or face BOC the perpendicular AD: draw, also, in that plane, the lines DH, DC, perpendicular to OB, OC, respectively; and join AH, AG; then will AH be perpendicular to OB, and AC to OC. It is evident, therefore, that the angles ACD, AHD, measure the angles between the planes AOC, COB, and AOB, COB, that is, the angles C and B of the spherical triangle ABC. It is also evident that the plane angles in O, are AOB $= c$, AOC $= b$, BOC $= a$. This being premised, the triangles ACO, ACD, the former right angled in C, the latter in D, give

$$\mathrm{AC} = \mathrm{AO} \sin \mathrm{AOC} = \mathrm{AO} \sin b$$
$$\mathrm{AD} = \mathrm{AC} \sin \mathrm{ACD} = \mathrm{AO} \sin b \sin \mathrm{C}.$$

In like manner, the triangles AOH, ADH, right angled in H and D, give

$$\mathrm{AH} = \mathrm{AO} \sin \mathrm{AOB} = \mathrm{AO} \sin c$$
$$\mathrm{AD} = \mathrm{AH} \sin \mathrm{AHD} = \mathrm{AO} \sin c \sin \mathrm{B}.$$



Making these two values of AD equal, we have $\mathrm{AO} \sin \mathrm{B} \sin c = \mathrm{AO} \sin \mathrm{C} \sin b$. And therefore, by division,

$$\frac{\sin \mathrm{B}}{\sin b} = \frac{\sin \mathrm{C}}{\sin c}.$$

In a similar manner it may be shown that

$$\frac{\sin \mathrm{A}}{\sin a} = \frac{\sin \mathrm{B}}{\sin b}: \text{ therefore,}$$

$$\frac{\sin \mathrm{A}}{\sin a} = \frac{\sin \mathrm{B}}{\sin b} = \frac{\sin \mathrm{C}}{\sin c} \ldots . (1.)$$

Hence, *the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

21. Draw CE and DF, respectively perpendicular and parallel to OB; then will the angle $\mathrm{DCF} = \mathrm{EOC} = a$. But the right angled triangles AOC, ACD, FCD, give

$$\mathrm{AC} = \mathrm{AO} \sin b, \ \mathrm{DC} = \mathrm{AC} \cos \mathrm{C} = \mathrm{AO} \sin b \cos \mathrm{C}$$
$$\text{and } \mathrm{FD} = \mathrm{DC} \sin a = \mathrm{AO} \sin a \sin b \cos \mathrm{C}.$$

Now $\mathrm{OH} = \mathrm{OE} + \mathrm{EH} = \mathrm{OE} + \mathrm{FD}$,
or $\mathrm{AO} \cos c = \mathrm{OC} \cos a + \mathrm{FD} = \mathrm{AO} \cos a \cos b + \mathrm{FD}$
$= \mathrm{AO} \cos a \cos b + \mathrm{AO} \sin a \sin b \cos \mathrm{C}$.

Therefore, dividing by AO, we have

$$\cos c = \cos a \cos b + \sin a \sin b \cos \mathrm{C}.$$

Similar relations are deducible for the other sides a and b: hence, generally

$$\left.\begin{array}{l} \cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A} \\ \cos b = \cos a \cos c + \sin a \sin c \cos \mathrm{B} \\ \cos c = \cos a \cos b + \sin a \sin b \cos \mathrm{C} \end{array}\right\} (2.)$$

22. These equations apply equally to the supplemental triangle. Thus, putting for the sides a, b, c,

$180° - A'$, $180° - B'$, $180° - C'$, &c. and for the angles A, B, C, $180° - a'$, $180° - b'$, &c. we shall have

$$-\cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a'.$$

Here again we have three symmetrical equations applying to *any* spherical triangle, viz.

$$\left.\begin{array}{l} \cos A = \cos a \sin B \sin C - \cos B \cos C \\ \cos B = \cos b \sin A \sin C - \cos A \cos C \\ \cos C = \cos c \sin A \sin B - \cos A \cos B \end{array}\right\} (3.)$$

23. Another important relation may be readily deduced. For, substituting for $\cos b$ in the third of the equations marked (2) its value in the second; substituting also for $\cos^2 a$ its value $1 - \sin^2 a$ (chap. i. 19), and striking out the common factor $\sin a$, we shall have

$$\cos c \sin a = \sin c \cos a \cos B + \sin b \cos C.$$

But, equa. (1) gives $\sin b = \dfrac{\sin B \sin c}{\sin C}$.

Hence, by substitution,

$$\cos c \sin a = \sin c \cos a \cos B + \frac{\sin B \cos C \sin c}{\sin C}.$$

Dividing by $\sin c$ we have,

$$\frac{\cos c}{\sin c} \sin a = \cos a \cos B + \frac{\sin B \cos C}{\sin C}.$$

But $\dfrac{\cos}{\sin} = \tan$ (chap. i. 19).

Therefore $\cot c \sin a = \cos a \cos B + \sin B \cot C$.

Thus, again, we get three symmetrical equations,

$$\left.\begin{array}{l} \cot a \sin b = \cos b \cos C + \sin C \cot A \\ \cot b \sin c = \cos c \cos A + \sin A \cot B \\ \cot c \sin a = \cos a \cos B + \sin B \cot C \end{array}\right\} (4.)$$

24. The classes of equations marked 1, 2, 3, 4, comprehend the whole of spherical trigonometry: or, in truth, the equations (2), from which the others may be made to flow, may be regarded as comprehending the whole. They require, however, some modifications to fit them for logarithmic computations, and become simplified in their application to some kinds of triangles. We shall, therefore, now show the pupil how they be-

come transformed when they are applied to the principal cases which occur in practice.

SECTION II.

Resolution of Right angled Spherical Triangles.

25. Suppose, in the first diagram in this chapter, the angle A to be right, or the faces OAB, OAC, to be perpendicular to each other. Then, since $\sin A = 1$, the equations marked (1) become

$$\frac{1}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \quad \text{Consequently,}$$

$$\left.\begin{array}{l} \sin B = \dfrac{\sin b}{\sin a} \ldots \sin C = \dfrac{\sin c}{\sin a} \ldots \\ \sin b = \sin B \sin a \ldots \sin c = \sin C \sin a \end{array}\right\} (5.)$$

Also, since $\cos A$ is then $= 0$, we have from equa. (2)

$$\cos a = \cos b \cos c \ldots . (6.)$$

For the same reason, the first of equa. (3) gives

$$\cos a \sin B \sin C = \cos B \cos C;$$

$$\text{whence } \cos a = \frac{\cos B \cos C}{\sin B \sin C} = \cot B \cot C \ldots . (7.)$$

Upon the same hypothesis, $\cot A$ becomes $= 0$, so that the first of equa. (4) becomes

$$\cot a \sin b = \cos b \cos C. \quad \text{Or, dividing by } \sin b,$$

$$\cot a = \frac{\cos b}{\sin b} \cos C = \cot b \cos C \ldots . (8.)$$

The two last of equa. (3) give also, upon the same hypothesis,

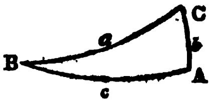
$$\left.\begin{array}{l} \cos B = \sin C \cos b \\ \cos C = \sin B \cos c \end{array}\right\} \ldots . (9.)$$

And, lastly, from equa. (4) we have

$$\left.\begin{array}{l} \cot B = \cot b \sin c \\ \cot C = \cot c \sin b \end{array}\right\} \ldots . (10.)$$

From these equations by a few obvious transformations, the *six* usual cases of right angled spherical triangles may be solved, as below.

Case I.



26. Given the hypothenuse a and an angle B; to find the rest;

$$\sin b = \sin a \sin B;$$

or, sin side req. = sin opp. angle × sin hypoth.

$$\tan c = \tan a \cos B;$$

or, tan side req. = tan hypoth. × cos included angle,

$$\cot C = \cos a \tan B;$$

or, cot angle req. = cos hypoth. × tan given angle.

In this case there can be nothing ambiguous; for, in applying the first form, it is known that the angle and the opposite side are always of the same affection; and in the two latter the rules for the changes of sines in the different quadrants (chap. iv. 9), will determine to which the result belongs.

Case II.

27. Given the hypothenuse a, and one of the sides b; to find the rest.

$$\sin B = \frac{\sin b}{\sin a}; \text{ or, sin angle req.} = \frac{\text{sin given side}}{\text{sin hypoth.}}.$$

$$\cos c = \frac{\cos a}{\cos b}; \text{ or, cos side req.} = \frac{\text{cos hypoth.}}{\text{cos given side}}.$$

$$\cos C = \tan b \cot a;$$

or, cos angle req. = tan given side × cot hypoth.

Case III.

28. Given the two sides including the right angle, namely, b and c; to find the rest.

$$\cos a = \cos b \cos c;$$

or, cos hypoth. = rectangle cosines of the sides.

$$\tan B = \frac{\tan b}{\sin c} \ldots \tan C = \frac{\tan c}{\tan b};$$

$$\text{or, tan angle req.} = \frac{\text{tan opp. side}}{\text{tan adj. side}}.$$

Case IV.

29. Given a side b, and its opposite angle B; to find the rest.

$$\sin a = \frac{\sin b}{\sin B}; \text{ or, } \sin \text{hypoth.} = \frac{\sin \text{given side}}{\sin \text{opp. angle}}.$$

$$\sin c = \tan b \cot B;$$

or, sin side req. = tan given side × cot opp. angle.

$$\sin C = \frac{\cos B}{\cos b}: \text{ or, } \sin \text{req. angle} = \frac{\cos \text{given angle}}{\cos \text{given side}}.$$

Case V.

30. Given a side c, and its adjacent angle B; to find the rest.

$$\tan b = \tan B \sin c;$$

or, tan side req. = tan opp. angle × sin given side.

$$\tan a = \frac{\tan c}{\cos B}; \text{ or, } \tan \text{hypoth.} = \frac{\tan \text{given side}}{\cos \text{given angle}}.$$

$$\text{or, } \cot a = \cos B \cot c;$$

that is, cot hypoth. = cos given angle × tan given side.

$$\cos C = \cos c \sin B;$$

or, cos angle req. = cos opp. side × sin given angle.

Case VI.

31. Given the two oblique angles B and C; to find the rest.

$$\cos c = \cot B \cot C;$$

or, cos hypoth. = rectangle cot's given angles.

$$\left.\begin{aligned} \cos b &= \frac{\cos B}{\sin C} \\ \cos c &= \frac{\cos C}{\sin B} \end{aligned}\right\} \cos \text{req. side} = \frac{\cos \text{opp. angle}}{\sin \text{adj. angle}}.$$

Note 1. Here the rule of the signs (chap. iv. 9) serves all along to determine the kind or affection of the unknown parts.

Note 2. In working by the logarithms, the student must observe, that when the resulting log. is the log. of a quotient, 10 must be *added* to the index: when it is the log. of a product, 10 must be *subtracted* from the index. This is done in conformity with the rule (chap. iv. 17), to make the terms homogeneous by multiplying or dividing by the powers of radius.

SECTION III.

Resolution of Oblique angled Spherical Triangles.

32. This may be effected by means of four general cases; each comprehending two or more problems.

Case I.

Given three of these four things, viz. two sides *b*, *c*, and their opposite angles B, C; to find the fourth.

This case comprehends two problems, in one of which the unknown quantity is an angle, in the other a side. They are both solved by means of equa. (1) of this chapter, from which we have

$$\sin C = \frac{\sin c \sin B}{\sin b} \ldots \sin c = \frac{\sin C \sin b}{\sin B}.$$

Case II.

33. Of these four things, viz. the three sides *a*, *b*, *c*, and an angle, any three being given, to find the fourth.

This case comprises three problems.

1. When the three sides are given, to find an angle. Here from equa. (2) we have

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \\ \cos B &= \frac{\cos b - \cos a \cos c}{\sin a \sin c}. \\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b}. \end{aligned}\right\}$$

But these are not fitted for logarithmic computation. Recurring, therefore, to (chap. iv. equa. U'), we have $1 + \cos \text{A} = 2\cos^2 \frac{1}{2}\text{A}$, and $1 - \cos \text{A} = 2 \sin^2 \frac{1}{2}\text{A}$. Hence,

$$2\cos^2 \tfrac{1}{2}\text{A} = \frac{\sin b \sin c}{\sin b \sin c} + \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a - \cos(b + c)}{\sin b \sin c}.$$

Hence also, $2 \sin^2 \frac{1}{2}\text{A} = \dfrac{\cos(b - c) - \cos a}{\sin b \sin c}$.

The latter of these divided by the former, gives,

$$\tan^2 \tfrac{1}{2}\text{A} = \frac{\cos(b - c) - \cos a}{\cos a - \cos(b + c)}.$$

Whence, from the 4th of equa. (U) chap. iv.

$$\tan^2 \tfrac{1}{2}\text{A} = \frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a + c - b)}{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a)}.$$

Hence we have, for the tangents of the half angles, these three symmetrical equations:

$$\left.\begin{aligned}
\tan \tfrac{1}{2}\text{A} &= \sqrt{\frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a + c - b)}{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a)}} \\
\tan \tfrac{1}{2}\text{B} &= \sqrt{\frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(b + c - a)}{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(a + c - b)}} \\
\tan \tfrac{1}{2}\text{C} &= \sqrt{\frac{\sin \frac{1}{2}(a + c - b) \sin \frac{1}{2}(b + c - a)}{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(a + b - c)}}
\end{aligned}\right\}(11.)$$

The expressions for the *sines* of the half angles might be deduced with equal facility. As they are symmetrical we shall put down but one, viz.

$$\sin \tfrac{1}{2}\text{A} = \sqrt{\frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a + c - b)}{\sin b \sin c}}.$$

Expressions for the cosines and cotangents of the half angles, may be readily found from the above, by the forms $\cos = \dfrac{\sin}{\tan}$, $\cot = \dfrac{\cos}{\sin}$.

Cor. When two of the sides, as b and c, become equal, the expression for $\sin \frac{1}{2}\mathrm{A}$, becomes $\sin \frac{1}{2}\mathrm{A} = \frac{\sin \frac{1}{2}a}{\sin b}$.

Cor. 2. If $a = b = c = 90°$, then $\sin \frac{1}{2}\mathrm{A} = \frac{\frac{1}{2}\sqrt{2}}{1}$ $= \frac{1}{2}\sqrt{2} = \sin 45°$; and $\mathrm{A} = \mathrm{B} = \mathrm{C} = 90°$.

Leaving other corollaries to be deduced by the student, let us proceed to the next problem in this case.

2. To find the side c opposite to the given angle C; that is, given two sides and the included angle, to find the third side.

Find from the data a dependent angle ϕ, such that

$$\tan \phi = \cos \mathrm{C} \tan b \ \ldots. (12.)$$

Substitute for $\cos \mathrm{C}$ in the third of equa. (2) its value in this, it will become

$$\cos c = \cos a \cos b + \sin a \cos b \tan \phi$$

$$= \cos b \left(\frac{\cos a \cos \phi + \sin a \sin \phi}{\cos \phi}\right);$$

$$\text{or, } \cos c = \frac{\cos b \cos (a - \phi)}{\cos \phi} \ \ldots. (13.)$$

Note. The equa. (12) obviously reduces to

$\frac{1}{\tan \phi} . \cos \mathrm{C} = \frac{1}{\tan b}$, or $\cot \phi \cos \mathrm{C} = \cot b$; which is analogous to equa. (8). So that b is the hypothenuse and ϕ one leg of a right angled triangle. The above transformation, therefore, is equivalent to the division of the proposed triangle into two, by an arc from the vertical angle A falling perpendicularly upon the opposite side a.

3. To find the side a, not opposite to the given angle; b, c, and C, being given.

Here find ϕ, as before, by equa. (12): then from equa. (13) we have

$$\cos (a - \phi) = \frac{\cos c \cos \phi}{\cos b} \ \ldots. (14.)$$

Hence a is known by adding ϕ.

Case III.

34. Of these four things, viz. two sides a and c, and two angles B and C, one opposite, the other adjacent; three being given, to find the fourth.

This case presents four problems.

1. Given a, c, B; to find C.

Determine an arc ϕ' by this condition,

$$\cot c = \cot \phi' \cos B, \text{ or } \frac{\cot c}{\cos B'} = \cot \phi' \ldots . (15.)$$

Substitute this value of cot c for it in the third of equa. (4); it will become

$$\sin a \cot \phi' \cos B = \cos a \cos B + \sin B \cot C;$$

$$\text{whence } \cot C = \frac{(\cot \phi' \sin a - \cos a) \cos B}{\sin B}$$

$$= (\cot \phi' \sin a - \cos a) \cot B;$$

$$\text{or, } \cot C = \frac{\cot B \sin (a - \phi')}{\sin \phi'} \ldots . (16.)$$

Note. The equa. (15) is akin to equa. (8); showing that the operation here performed is equivalent to letting fall a perpendicular arc from the angle A to the base a; the subsidiary arc ϕ' being the segment adjacent to the angle B.

2. Given B, C, c; to find a.

Here ϕ' must be found by equa. (15), and then from equa. (16) we have

$$\sin (a - \phi') = \frac{\cot C \sin \phi'}{\cot B} \ldots . (17.)$$

Whence a becomes known.

3. Given B, C, a; to find c.

Find a dependent angle ϕ'' by making

$$\cot C = \cos a \tan \phi'', \text{ or } \frac{\cot C}{\cos a} = \tan \phi'' \ldots . (18.)$$

Substitute this value of cot C for it in the third of equa. (4), and it will reduce to

$$\cot c = \frac{\cot a \cos (B - \phi'')}{\cos \phi''} \ldots . (19.)$$

4. Given a, c, C; to find B.

Determine φ'' from equa. (18); then

$$\cos(\mathrm{B} - \varphi'') = \frac{\cot c \cos \varphi''}{\cot a} \ldots\ldots (20.)$$

From which B becomes known.

Case IV.

35. Of these four things, viz. the three angles and a side, suppose c, any three being given, to find the fourth.

This comprises three problems.

1. Given the three angles; to find a side.

Suppose the first of equa. (11) to be applied to the solution of the supplemental triangle, by changing a, b, c, and C, into a', b', c', and C$'$. Then, to bring it back to the triangle proposed, let there be substituted for a', b', c', and C$'$, the corresponding values 180° − A, 180° − B, 180° − C, 180° − c. Those equations will then be transformed into the following, applicable to the present problem.

$$\left.\begin{aligned}
\cot \tfrac{1}{2}a &= \sqrt{\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C})\cos\frac{1}{2}(\mathrm{A}-\mathrm{B}+\mathrm{C})}{-\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}+\mathrm{A})\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})}}\\
\cot \tfrac{1}{2}b &= \sqrt{\frac{\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C})}{-\cos\frac{1}{2}(\mathrm{A}+\mathrm{C}+\mathrm{B})\cos\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B})}}\\
\cot \tfrac{1}{2}c &= \sqrt{\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B})\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})}{-\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C})}}
\end{aligned}\right\}(21.)$$

The following are the expressions for the sines of the half angles, viz.

$$\left.\begin{aligned}
\sin \tfrac{1}{2}a &= \sqrt{\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\cos\frac{1}{2}(\mathrm{B}+\mathrm{C}-\mathrm{A})}{-\sin\mathrm{B}\sin\mathrm{C}}}\\
\sin \tfrac{1}{2}b &= \sqrt{\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\cos\frac{1}{2}(\mathrm{A}+\mathrm{C}-\mathrm{B})}{-\sin\mathrm{A}\sin\mathrm{C}}}\\
\sin \tfrac{1}{2}c &= \sqrt{\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C})}{-\sin\mathrm{A}\sin\mathrm{B}}}
\end{aligned}\right\}(22.)$$

Note. In these expressions, although the denominators are negative, the whole fractions under the radical are always positive.

The expressions for the tangents and cosines are omitted, to save room,

2. Given A, B, c; to find C.

Here, by applying in like manner, the equations (12), (13), to the supplemental triangle, we shall have

$$\cot \phi = \cos c \tan B \ldots. (23.)$$

from which the subsidiary angle ϕ may be determined; and thence

$$\cos C = \frac{\cos B \sin (A - \phi)}{\sin \phi} \ldots. (24.)$$

3. Given B, C, c: to find A.

Find ϕ from equa. (23); then from equa. (24) there results,

$$\sin (A - \phi) = \frac{\cos C \sin \phi}{\cos B} \ldots. (25.)$$

from which A is known.

SECTION IV.

On the Analogies of Napier.

36. These are four simple and elegant formulæ discovered by the celebrated inventor of logarithms, of which two serve to determine any two angles of a spherical triangle, by means of the two opposite sides and their included angle; while the other two serve to determine any two sides, by means of the opposite angles and their contained side. Thus, therefore, they together with equa. (1), will serve for the solution of *all* the cases of oblique spherical triangles. The investigation of these analogies may be given, as below.

If from the first of equa. (2), cos c be exterminated, there will result, after a little reduction,

$$\cos A \sin c = \cos a \sin b - \cos C \sin a \cos b:$$

and by a simple permutation of letters,

$$\cos B \sin c = \cos b \sin a - \cos C \sin b \cos a:$$

adding these equations together, and reducing, we have

$$\sin c\,(\cos A + \cos B) = (1 - \cos C) \sin (a + b).$$

Now we have from equa. (1)

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Freeing these equations from their denominators, and respectively adding and subtracting them, there results,

$$\sin c\,(\sin A + \sin B) = \sin C\,(\sin a + \sin b),$$
$$\text{and } \sin c\,(\sin A - \sin B) = \sin C\,(\sin a - \sin b).$$

Dividing each of these two equations by the preceding, there will be obtained,

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos C} \cdot \frac{\sin a + \sin b}{\sin (a + b)}$$
$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin C}{1 - \cos C} \cdot \frac{\sin a - \sin b}{\sin (a + b)}.$$

Comparing these with the equations in arts. 24, 25, 26, chap. iv. we shall have

$$\left.\begin{aligned}\tan \tfrac{1}{2}(A + B) &= \cot \tfrac{1}{2}C \cdot \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \\ \tan \tfrac{1}{2}(A - B) &= \cot \tfrac{1}{2}C \cdot \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}\end{aligned}\right\} (26.)$$

The above equations expressed as analogies, are

$$\cos \tfrac{1}{2}(a + b) : \cos \tfrac{1}{2}(a - b) :: \cot \tfrac{1}{2}C : \tan \tfrac{1}{2}(A + B)$$
$$\sin \tfrac{1}{2}(a + b) : \sin \tfrac{1}{2}(a - b) :: \cot \tfrac{1}{2}C : \tan \tfrac{1}{2}(A - B).$$

These analogies being applied to the supplemental triangle, by putting 180° − A, 180° − B, &c. for *a*, *b*, &c. we have

$$\cos \tfrac{1}{2}(A + B) : \cos \tfrac{1}{2}(A - B) :: \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(a + b)$$
$$\sin \tfrac{1}{2}(A + B) : \sin \tfrac{1}{2}(A - B) :: \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(a - b).$$

37. From a due consideration of the four analogies it results,

1st. That $\frac{1}{2}(A - B) < 90°$, or that the difference of two *angles* of a spherical triangle is less than 180°.

2dly. That $\frac{1}{2}(a + b)$ and $\frac{1}{2}(A + B)$ are always of the same affection.

3dly. That the difference of two *sides* is always less than 180°.

4thly. That $(a - b)$ and $(A - B)$ have always the same sign; whence it follows, that the greatest angle is opposite to the greatest side, and reciprocally.

To these it may be added,

5thly. That the least angle is opposite to the least side, and the mean angle, to the mean side.

One or other of these observations will serve to remove the ambiguity in the doubtful cases, where either a, b, and B, or A, B, and b, are given.

38. We may now collect the most commodious theorems, and present in one place all that will be usually required in the solution of oblique angled spherical triangles.

$$1.\quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

$$2.\quad \tan \tfrac{1}{2}A = \sqrt{\frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a + c - b)}{\sin \frac{1}{2}(b + c - a) \sin \frac{1}{2}(a + b + c)}}$$

$$3.\quad \tan \tfrac{1}{2}B = \sqrt{\frac{\sin \frac{1}{2}(b + c - a) \sin \frac{1}{2}(a + b - c)}{\sin \frac{1}{2}(a + c - b) \sin \frac{1}{2}(a + b + c)}}$$

$$4.\quad \tan \tfrac{1}{2}C = \sqrt{\frac{\sin \frac{1}{2}(a + c - b) \sin \frac{1}{2}(b + c - a)}{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a + b + c)}}$$

$$5.\quad \tan \tfrac{1}{2}a = \sqrt{\frac{-\cos \frac{1}{2}(B + C - A) \cos \frac{1}{2}(A + B + C)}{\cos \frac{1}{2}(A + B - C) \cos \frac{1}{2}(A + C - B)}}$$

$$6.\quad \tan \tfrac{1}{2}b = \sqrt{\frac{-\cos \frac{1}{2}(A + C - B) \cos \frac{1}{2}(A + R + C)}{\cos \frac{1}{2}(B + C - A) \cos \frac{1}{2}(A + B - C)}}$$

$$7.\quad \tan \tfrac{1}{2}c = \sqrt{\frac{-\cos \frac{1}{2}(A + B - C) \cos \frac{1}{2}(A + B - C)}{\cos \frac{1}{2}(A + C - B) \cos \frac{1}{2}(B + C - A)}}$$

$$8.\quad \tan \frac{b - a}{2} = \tan \tfrac{1}{2}c \, \frac{\sin \frac{1}{2}(B - A)}{\sin \frac{1}{2}(B + A)}$$

$$9.\quad \tan \frac{b + a}{2} = \tan \tfrac{1}{2}c \, \frac{\cos \frac{1}{2}(B - A)}{\cos \frac{1}{2}(B + A)}$$

10. $\tan\frac{c-b}{2} = \tan\frac{1}{2}a\,\frac{\sin\frac{1}{2}(C-B)}{\sin\frac{1}{2}(C+B)}$

11. $\tan\frac{c+b}{2} = \tan\frac{1}{2}a\,\frac{\cos\frac{1}{2}(C-B)}{\cos\frac{1}{2}(C+B)}$

12. $\tan\frac{a-c}{2} = \tan\frac{1}{2}b\,\frac{\sin\frac{1}{2}(A-C)}{\sin\frac{1}{2}(A+C)}$

13. $\tan\frac{a+c}{2} = \tan\frac{1}{2}b\,\frac{\cos\frac{1}{2}(A-C)}{\cos\frac{1}{2}(A+C)}$

14. $\tan\frac{B-A}{2} = \cot\frac{1}{2}C\,\frac{\sin\frac{1}{2}(b-a)}{\sin\frac{1}{2}(b+a)}$

15. $\tan\frac{B+A}{2} = \cot\frac{1}{2}C\,\frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}$

16. $\tan\frac{C-B}{2} = \cot\frac{1}{2}A\,\frac{\sin\frac{1}{2}(c-b)}{\sin\frac{1}{2}(c+b)}$

17. $\tan\frac{C+B}{2} = \cot\frac{1}{2}A\,\frac{\cos\frac{1}{2}(c-b)}{\cos\frac{1}{2}(c+b)}$

18. $\tan\frac{A-C}{2} = \cot\frac{1}{2}B\,\frac{\sin\frac{1}{2}(a-c)}{\sin\frac{1}{2}(a+c)}$

19. $\tan\frac{A+C}{2} = \cot\frac{1}{2}B\,\frac{\cos\frac{1}{2}(a-c)}{\cos\frac{1}{2}(a+c)}$

20. $\tan\frac{1}{2}c = \tan\frac{1}{2}(b-a)\,\frac{\sin\frac{1}{2}(B+A)}{\sin\frac{1}{2}(B-A)}$

21. $\tan\frac{1}{2}c = \tan\frac{1}{2}(b+a)\,\frac{\cos\frac{1}{2}(B+A)}{\cos\frac{1}{2}(B-A)}$

22. $\tan\frac{1}{2}a = \tan\frac{1}{2}(c-b)\,\frac{\sin\frac{1}{2}(C+B)}{\sin\frac{1}{2}(C-B)}$

23. $\tan\frac{1}{2}a = \tan\frac{1}{2}(c+b)\,\frac{\cos\frac{1}{2}(C+B)}{\cos\frac{1}{2}(C-B)}$

24. $\tan\frac{1}{2}b = \tan\frac{1}{2}(a-c)\,\frac{\sin\frac{1}{2}(A+C)}{\sin\frac{1}{2}(A-C)}$

25. $\tan\frac{1}{2}b = \tan\frac{1}{2}(a+c)\,\frac{\cos\frac{1}{2}(A+C)}{\cos\frac{1}{2}(A-C)}$

26. $\cot \frac{1}{2}C = \tan \frac{1}{2}(B - A) \frac{\sin \frac{1}{2}(b + a)}{\sin \frac{1}{2}(b - a)}$

27. $\cot \frac{1}{2}C = \tan \frac{1}{2}(B + A) \frac{\cos \frac{1}{2}(b + a)}{\cos \frac{1}{2}(b - a)}$

28. $\cot \frac{1}{2}A = \tan \frac{1}{2}(C - B) \frac{\sin \frac{1}{2}(c + b)}{\sin \frac{1}{2}(c - b)}$

29. $\cot \frac{1}{2}A = \tan \frac{1}{2}(C + B) \frac{\cos \frac{1}{2}(c + b)}{\cos \frac{1}{2}(c - b)}$

30. $\cot \frac{1}{2}B = \tan \frac{1}{2}(A - C) \frac{\sin \frac{1}{2}(a + c)}{\sin \frac{1}{2}(a - c)}$

31. $\cot \frac{1}{2}B = \tan \frac{1}{2}(A + C) \frac{\cos \frac{1}{2}(a + c)}{\cos \frac{1}{2}(a - c)}$

SECTION V.

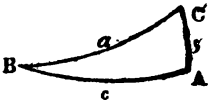
Mnemonics of Spherical Trigonometry.

39. As it is, obviously, difficult to retain in recollection the necessary rules and formulæ in spherical trigonometry, attempts have been made by different mathematicians, to assist the student by contrivances akin to those which occur in repositories of artificial memory.

Napier, to whom this department of science is so much indebted, at the end of his *Mirifici Canonis Constructio*, obscurely suggested a simple and comprehensive method, characterised by the name of Napier's *Rules for the Circular Parts;* which apply, 1st, to *right angled* spherical triangles; 2dly, by means of the polar triangle, to *quadrantal* triangles; and, 3dly, by means of a perpendicular from the vertical angle, to *oblique* angled spherical triangles. These rules were developed much more perspicuously by Gellibrand; and have, since his time, been explained by almost every writer on spherical trigonometry.

40. In a spherical triangle ABC, right angled at A, we have, as was shown in arts. 25—31, of this chapter,

$\sin c = \sin a \sin \mathrm{C} = \tan b \cot \mathrm{B}$
$\sin b = \sin a \sin \mathrm{B} = \tan c \cot \mathrm{C}$
$\cos a = \cos c \cos b = \cot \mathrm{B} \cot \mathrm{C}$
$\cos \mathrm{B} = \cos b \sin \mathrm{C} = \tan c \cot a$
$\cos \mathrm{C} = \cos c \sin \mathrm{B} = \cot a \tan b$



These equations include only *five* of the six parts of the triangle, the sixth part, viz. the angle A being constant. Now, Napier's circular parts are, the complements of the angles B and C, the complement of the hypothenuse a, and the other two sides b and c: that is, $90° - a$, $90° - \mathrm{B}$, $90° - \mathrm{C}$, b, and c, are the five *circular parts*. Any one of these five may be called a *middle part*; then the two parts next adjacent, one on the right, the other on the left, [not including the right angle] are called *adjacent* parts; and the next two, each separated from the middle part by an adjacent part, are called *opposite* parts. This premised, the rules are,

1. The rectangle of the radius and the sine of the middle part = rectangle of the tangents of the adjacent parts.

2. The rectangle of the radius and the sine of the middle part = rectangle of the cosines of the opposite parts.

These rules may be comprehended in one expression, extremely easy to remember, simply remarking that the vowels in the contractions *sin*, *tan*, *cos*, are respectively the same as those in the first syllable of *mid*dle, *ad*jacent, *op*posite: for then, regarding unity as radius, we shall have

sin mid = rect tan adja = rect cos op.

A rule which, by taking a, b, c, &c. successively for the middle part, will be found to comprise the five expressions given above.

41. By the circular parts of an *oblique* spherical triangle are meant its three sides, and the supplements of its three angles. Any of these six being assumed as a middle part, the opposite parts are those two that are

of the same denomination with it; that is, if the middle part is one of the sides, the opposite parts are the other two; and if the middle part is the supplement of one of the angles, the opposite parts are the supplements of the other two.

42. Mr. Walter Fisher has also given, in the Transactions of the Royal Society of Edinburgh, rules of easy recollection which will serve for the solution of all the cases of plane and spherical triangles. They are included in four theorems, which may be applied to plane triangles by taking, instead of the sine or tangent of a side, the side itself.

Theor. 1. Given two parts, and an opposite one.

$$\mathrm{S}.\mathrm{A} : \mathrm{S}.\mathrm{O} :: \mathrm{S}.a : \mathrm{S}.o,$$

Theor. 2. An included part given, or sought.

$$\mathrm{S}.\frac{\mathrm{A}-a}{2} : \mathrm{S}.\frac{\mathrm{A}+a}{2} :: \mathrm{T}.\frac{\mathrm{O}-o}{2} : \mathrm{T}.\frac{\mathrm{M}}{2}.$$

Theor. 3. $\mathrm{T}.\frac{\mathrm{A}-a}{2} : \mathrm{T}.\frac{\mathrm{A}+a}{2} :: \mathrm{T}.\frac{\mathrm{O}-o}{2} : \mathrm{T}.\frac{\mathrm{O}+o}{2}.$

Theor. 4. Given the three sides or angles of an oblique angled triangle.

$$\mathrm{S}.\mathrm{A} \times \mathrm{S}.a : \mathrm{R}^2 :: \mathrm{S}.\frac{(\mathrm{A}+a)+l}{2} \times \mathrm{S}.\frac{(\mathrm{A}+a)-l}{2} : \mathrm{S}^2.\frac{\mathrm{M}}{2}.$$

Here M denotes the middle part of the triangle, and must always be assumed *between* two given parts. It is either a side or the *supplement* of an angle; and is sometimes given, sometimes not.

A and *a* are the two parts *adjacent* to the middle, and of a different denomination from it.

O and *o* denote the two parts *opposite* to the adjacent parts, and of the *same* denomination with the middle part.

l is the *last*, or most distant part, and of a different denomination from the middle part.

That these four theorems may be called to mind with the greater facility, the following four words formed by abbreviating the terms of the respective analogies should be committed to memory, viz.

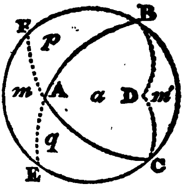
Sao, satom, tao, sarsalm.

SECTION VI.

On the Areas of Spherical Triangles and Polygons, and the Measures of Solid Angles.

Theorem I.

43. In every spherical triangle, the following proportion obtains, viz. as four right angles, (or 360°), to the surface of a hemisphere; or, as two right angles (or 180°), to a great circle of the sphere; so is the excess of the three angles of the triangle, above two right angles, to the area of the triangle.



Let ABC be the spherical triangle. Complete one of its sides, as BC, into the circle BCEF, which may be supposed to bound the upper hemisphere. Prolong also, at both ends, the two sides AB, AC, until they form semicircles estimated from each angle, that is, until BAE = ABD = CAF = ACD = 180°. Then will CBF = 180° = BFE; and consequently the triangle AEF, on the anterior hemisphere, will be equal to the triangle BCD on the opposite hemisphere. Putting m, m', to represent the surface of these triangles, p for that of the triangle BAF, q for that of CAE, and a for that of the proposed triangle ABC. Then a and m' together (or their equal a and m together) make up the surface of a spheric lune comprehended between the two semicircles ACD, ABD, inclined in the angle A: a and p together make up the lune included between the semicircles CAF, CBF, making the angle C: a and q together make up the spheric lune included between the semicircles BCE, BAE, making the angle B. And the surface of each of these lunes, is to that of the hemisphere, as the angle made by the comprehending semicircles, to two right angles. Therefore, putting $\frac{1}{2}$s for the surface of the hemisphere, we have

$$180^\circ : \text{A} :: \tfrac{1}{2}s : a + m.$$
$$180^\circ : \text{B} :: \tfrac{1}{2}s : a + q.$$
$$180^\circ : \text{C} :: \tfrac{1}{2}s : a + p.$$

Whence, $180^\circ : \text{A} + \text{B} + \text{C} :: \frac{1}{2}s : 3a + m + p + q = 2a + \frac{1}{2}s$; and consequently, by division of proportion, as $180^\circ : \text{A} + \text{B} + \text{C} - 180^\circ :: \frac{1}{2}s : 2a + \frac{1}{2}s - \frac{1}{2}s = 2a$;

or, $180^\circ : \text{A} + \text{B} + \text{C} - 180^\circ :: \frac{1}{4}s : a = \frac{1}{2}s \,.\, \dfrac{\text{A}+\text{B}+\text{C}-180^\circ}{360^\circ}$.

Q. E. D.*

Cor. 1. Hence the excess of the three angles of any spherical triangle above two right angles, termed technically the *spherical excess*, furnishes a correct measure of the surface of that triangle.

Cor. 2. If $\pi = 3{\cdot}141593$, and d the diameter of the sphere, then is $\pi d^2 \,.\, \dfrac{\text{A} + \text{B} + \text{C} - 180^\circ}{720^\circ}$ = the area of the spherical triangle.

Cor. 3. Since the length of the radius, in any circle, is equal to the length of 57·2957795 degrees, measured on the circumference of that circle; if the *spherical excess* be multiplied by 57·2957795, the product will express the surface of the triangle in square degrees.

Cor. 4. When $a = 0$, then $\text{A} + \text{B} + \text{C} = 180^\circ$: and when $a = \frac{1}{2}s$, then $\text{A} + \text{B} + \text{C} = 540^\circ$. Consequently the sum of the three angles of a spherical triangle, is always between 2 and 6 right angles: which is another confirmation of art. 19, p. 83.

---

* This determination of the area of a spherical triangle is due to *Albert Girard* (who died about 1633). But the demonstration now commonly given of the rule was first published by Dr. Wallis. It was considered as a mere speculative truth, until General Roy, in 1787, employed it very judiciously in the great Trigonometrical Survey, to correct the errors of spherical angles. See Phil. Trans. vol. 80. and chap. xii. of this volume.

Cor. 5. When *two* of the angles of a spherical triangle are right angles, the surface of the triangle varies with its third angle. And when a spherical triangle has *three* right angles its surface is one-eighth of the surface of the sphere.

Remark. The mode of finding the spherical excess, and thence the area when the three angles of a spherical triangle are given, is obvious enough; but it is often requisite to ascertain it by means of other data, as, when two sides and the included angle are given, or when all the three sides are given. In the former case, let a and b be the two sides, C the included angle, and E the spherical excess: then is $\cot \frac{1}{2}\mathrm{E} = \frac{\cot \frac{1}{2}a \,.\, \cot \frac{1}{2}b + \cos \mathrm{C}}{\sin \mathrm{C}}$. When the three sides a, b, c, are given, the spherical excess may be found by the following very elegant theorem, discovered by Simon Lhuillier:

$$\tan \tfrac{1}{4}\mathrm{E} = \sqrt{}\left(\tan \frac{a+b+c}{4} \,.\, \tan \frac{a+b-c}{4} \,.\, \tan \frac{a-b+c}{4} \right.$$
$$\left. .\, \tan \frac{-a+b+e}{4}\right).$$

The investigation of these theorems would occupy more space than can be allotted to them in the present volume.

Theorem II.

44. In every spherical polygon, or surface included by any number of intersecting great circles, the subjoined proportion obtains, viz. as four right angles, or 360°, to the surface of a hemisphere; or, as two right angles, or 180°, to a great circle of the sphere; so is the excess of the sum of the angles above the product of 180° and two less than the number of angles of the spherical polygon, to its area.

For, if the polygon be supposed to be divided into as many triangles as it has sides, by great circles drawn from all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by theor. 1, it will be as $360° : \frac{1}{2}s :: A + B + C - 180°$: its area. Therefore, putting P for the sum of all the angles of the polygon, n for their number, and V for the sum of all the vertical angles of its constituent triangles, it will be, by composition,

as $360° : \frac{1}{2}s :: P + V - 180° . n$: surface of the polygon. But V is manifestly equal to 360° or $180° \times 2$. Therefore,

as $360° : \frac{1}{2}s :: P - (n - 2)\ 180° : \frac{1}{2}s . \frac{P - (n - 2)\ 180°}{360°}$, the area of the polygon. Q. E. D.

Cor. 1. If π and d represent the same quantities as in theor. 1, cor. 2, then the surface of the polygon will be expressed by $\pi d^2 . \frac{P - (n - 2)\ 180°}{720°}$.

Cor. 2. If $R° = 57{\cdot}2957795$, then will the surface of the polygon in square degrees be $= R° . [P - (n - 2)\ 180°]$.

Cor. 3. When the surface of the polygon is 0, then $P = (n - 2)\ 180°$; and when it is a maximum, that is, when it is equal to the surface of the hemisphere, then $P = (n - 2)\ 180° + 360° = n . 180°$: consequently P, the sum of all the angles of any spheric polygon, is always *less* than $2n$ right angles, but *greater* than $(2n - 4)$ right angles, n denoting the number of angles of the polygon.

Nature and Measure of Solid Angles.

45. A *solid angle* is defined by Euclid, that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

Others define it the angular space comprised between several planes meeting in one point.

It may be defined still more generally, the *angular space* included between several plane surfaces or one or more curved surfaces, meeting in the point which forms the summit of the angle.

According to this definition, solid angles bear just the same relation to the surfaces which comprise them, as plane angles do to the lines by which they are included: so that, as in the latter, it is not the magnitude of the lines, but their mutual inclination, which determines the angle; just so, in the former, it is not the magnitude of the planes, but *their* mutual inclinations, which determine the angles. And hence all those geometers, from the time of Euclid down to the present period, who have confined their attention principally to the magnitude of the plane angles, instead of their relative positions, have never been able to develope the properties of this class of geometrical quantities; but have affirmed that no solid angle can be said to be the half or the double of another, and have spoken of the bisection and trisection of solid angles, even in the simplest cases, as impossible problems.

But all this supposed difficulty vanishes, and the doctrine of solid angles becomes simple, satisfactory, and universal in its application, by assuming *spherical surfaces* for their measure; just as circular arcs are assumed for the measures of plane angles.* Imagine, that from the summit of a solid angle (formed by the meeting of three planes) as a centre, any sphere be described, and that those planes are produced till they cut the surface of the sphere; then will the surface of the spherical triangle, included between those planes, be a proper measure of the solid angle made by the planes at their common point of meeting: for no change can be conceived in the relative position of those planes, that is, in the magnitude of the solid angle, without a corresponding and proportional mutation in the surface of the spherical triangle. If, in like manner, the three or more surfaces, which by their meeting constitute another solid angle, be produced till they cut the surface of the same or an equal sphere, whose centre coincides with the summit of the angle; the surface of the spheric triangle or polygon, included between the planes which determine the angle, will be a correct measure of *that* angle. And the ratio which subsists between the areas of the spheric triangles, polygons, or other surfaces thus formed, will be accurately the ratio which subsists between the solid angles, constituted by the meeting of the several planes or surfaces, at the centre of the sphere.

---

* This disquisition on solid angles was first published in the 3d volume of Dr. Hutton's Course of Mathematics, in the year 1811. At that time I thought the notion of measuring this class of geometrical magnitudes by means of spherical triangles and polygons was, though extremely obvious and natural, perfectly new. I have since found, by consulting Montucla's History of Mathematics, vol. ii. p. 8, that Albert Girard, in his *Invention nouvelle en Algèbre*, advanced an analogous theory.

While I am adverting to the third volume of Dr. Hutton's Course, I beg to mention, in order to account for any instances of close correspondence which may be found between parts of this volume and of that (though they, I believe, will occur quite as seldom as can well be expected when the same person is treating of the same subjects), that the 2d, 3d, 4th, 5th, 6th, 7th, 8th, 9th, and 11th chapters of that volume were composed by me; the remainder by my excellent friend the author of that Course.

Hence, the comparison of solid angles becomes a matter of great ease and simplicity: for, since the areas of spherical triangles are measured by the excess of the sums of their angles each above two right angles (theor. 1); and the areas of spherical polygons of n sides, by the excess of the sum of their angles above $2n - 4$ right angles (theor. 2); it follows, that the magnitude of a trilateral solid angle, will be measured by the excess of the sum of the three angles, made respectively by its bounding planes, above two right angles; and the magnitudes of solid angles formed by n bounding planes, by the excess of the sum of the angles of inclination of the several planes above $2n - 4$ right angles.

As to solid angles limited by curve surfaces, such as the angles at the vertices of cones; they will manifestly be measured by the spheric surfaces cut off by the prolongation of their bounding surfaces, in the same manner as angles determined by planes are measured by the triangles or polygons, they mark out upon the same, or an equal sphere. In all cases, the maximum limit of solid angles, will be the *plane* towards which the various planes determining such angles approach, as they diverge further from each other about the same summit; just as a right line is the maximum limit of plane angles, being formed by the two bounding lines when they make an angle of 180°. The maximum limit of solid angles is measured by the surface of a hemisphere, in like manner as the maximum limit of plane angles is measured by the arc of a semicircle. The solid right angle (either angle, for example, of a cube) is $\frac{1}{4}$ $(= \frac{1}{2}^2)$ of the maximum solid angle: while the plane right angle is half the maximum plane angle.

The analogy between plane and solid angles being thus traced, we may proceed to exemplify this theory by a few instances; assuming 1000 as the numeral measure of the maximum solid angle $= 4$ times 90° solid $= 360^\circ$ solid.

1. The solid angles of right prisms are compared with great facility. For, of the three angles made by the three planes which, by their meeting, constitute every such solid angle, two are right angles; and the third is the same as the corresponding plane angle of the polygonal base; on which, therefore, the measure of the solid angle depends. Thus, with respect to the right prism with an equilateral triangular base, each solid angle is formed by planes which respectively make angles of 90°, 90°, and 60°. Consequently $90^\circ + 90^\circ + 60^\circ - 180^\circ = 60^\circ$, is the measure of such angle, compared with 360° the maximum angle. It is, therefore, one-sixth of the maximum angle. A right prism with a square base has, in like manner, each solid angle measured by $90^\circ + 90^\circ + 90^\circ - 180^\circ = 90^\circ$, which is $\frac{1}{4}$ of the maximum angle. And thus it may be found, that each solid angle of a right prism, with an equilateral

triangular base is $\frac{1}{6}$ max. angle $= \frac{1}{6} \cdot 1000$.
square base is $\frac{1}{4}$ .... $= \frac{2}{8} \cdot 1000$.
pentagonal base is .... $= \frac{3}{10} \cdot 1000$.
hexagonal is $\frac{1}{3}$ .... $= \frac{4}{12} \cdot 1000$.
heptagonal is .... $= \frac{5}{14} \cdot 1000$.
octagonal is $\frac{3}{8}$ .... $= \frac{6}{16} \cdot 1000$.
nonagonal is .... $= \frac{7}{18} \cdot 1000$.
decagonal is $\frac{2}{5}$ .... $= \frac{8}{20} \cdot 1000$.
undecagonal is .... $= \frac{9}{22} \cdot 1000$.
duodecagonal is $\frac{5}{12}$ .... $= \frac{10}{24} \cdot 1000$.
m gonal is .... $= \frac{m-2}{2m} \cdot 1000$.

Hence it may be deduced, that each solid angle of a regular prism, with triangular base, is *half* each solid angle of a prism with a regular hexagonal base. Each with regular

square base $= \frac{2}{3}$ of each, with regular octagonal base,
pentagonal $= \frac{3}{4}$ .... decagonal,
hexagonal $= \frac{4}{5}$ .... duodecagonal,
$\frac{1}{2}m$ gonal $= \frac{m-4}{m-2}$ .... m gonal base.

Hence again we may infer, that the sum of all the solid angles of any prism of triangular base, whether that base be regular or irregular, is *half* the sum of the solid angles of a prism of quadrangular base, regular or irregular. And, the sum of the solid angles of any prism of

tetragonal base is $= \frac{2}{3}$ sum of ang. in prism of pentag. base,
pentagonal .... $= \frac{3}{4}$ .... hexagonal,
hexagonal .... $= \frac{4}{5}$ .... heptagonal,
m gonal ...... $= \frac{m-2}{m-1}$ .... $(m+1)$ gonal.

2. Let us compare the solid angles of the five regular bodies. In these bodies, if m be the number of sides of each face; n the number of planes which meet at each solid angle; $\frac{1}{2}\bigcirc$ = half the circumference or 180°; and A the plane angle made by two adjacent faces;

then we have $\sin \frac{1}{2}A = \dfrac{\cos \frac{1}{2n} \bigcirc}{\sin \frac{1}{2m} \bigcirc}$. This theorem gives, for the plane angle formed by every two contiguous faces of the tetraëdron, 70° 31′ 42″; of the hexaëdron, 90°; of the octaëdron, 109° 28′ 18″; of the dodecaëdron, 116° 33′ 54″; of the icosaëdron, 138° 11′ 23″. But, in these polyedræ, the number of faces meeting about each solid angle, are 3, 3, 4, 3, 5, respectively. Consequently the several solid angles will be determined by the subjoined proportions:

Solid Angle.

$360^\circ : 3.70^\circ 31' 42'' - 180^\circ :: 1000 : 87{\cdot}73611$ tetraëdron.
$360^\circ : 3.90^\circ - 180^\circ :: 1000 : 250{\cdot}$ hexaëdron.
$360^\circ : 4.109^\circ 28' 18'' - 360^\circ :: 1000 : 216{\cdot}35185$ octaëdron.
$360^\circ : 3.116^\circ 33' 54'' - 180^\circ :: 1000 : 471{\cdot}395$ dodecaëdron.
$360^\circ : 5.138^\circ 11' 23'' - 540^\circ :: 1000 : 419{\cdot}30169$ icosaëdron.

3. The solid angles at the vertices of cones, will be determined by means of the spheric segments cut off at the bases of those cones; that is, if right cones, instead of having plane bases, had bases formed of the segments of equal spheres, whose centres were the vertices of the cones, the surfaces of those segments would be measures of the solid angles at the respective vertices. Now, the surfaces of spheric segments, are to the surface of the hemisphere, as their altitudes, to the radius of the sphere; and, therefore, the solid angles at the vertices of right cones will be to the maximum solid angle, as the excess of the slant side above the axis of the cone, to the slant side of the cone. Thus, if we wish to ascertain the solid angles at the vertices of the equilateral and the right angled cones; the axis of the former is $\frac{1}{2}\sqrt{3}$, of the latter, $\frac{1}{2}\sqrt{2}$, the slant side of each being unity. Hence,

Angle at vertex.

$1 : 1 - \frac{1}{2}\sqrt{3} :: 1000 : 133{\cdot}97464$, equilateral cone,
$1 : 1 - \frac{1}{2}\sqrt{2} :: 1000 : 292{\cdot}89322$, right angled cone.

4. From what has been said, the mode of determining the solid angles at the vertices of pyramids will be sufficiently obvious. If the pyramids be regular ones, if N be the number of faces meeting about the vertical angle in one, and A the angle of inclination of each two of its plane faces; if n be the number of planes meeting about the vertex of the other, and a the angle of inclination of each two of its faces: then will the vertical angle of the former, be to the vertical angle of the latter pyramid, as $\text{NA} - (\text{N} - 2)\,180°$, to $na - (n - 2)\,180°$.

If a cube be cut by diagonal planes, into six equal pyramids with square bases, their vertices all meeting at the centre of the circumscribing sphere; then each of the solid angles, made by the four planes meeting at each vertex, will be $\frac{1}{3}$ of the maximum solid angle; and each of the solid angles at the bases of the pyramids, will be $\frac{1}{12}$ of the maximum solid angle. Therefore, each solid angle at the base of such pyramid, is *one-fourth* of the solid angle at its vertex: and, if the angle at the vertex be bisected, as described below, either of the solid angles arising from the bisection, will be double of either solid angle at the base. Hence also, and from the first subdivision of this inquiry, each solid angle of a prism, with equilateral triangular base, will be *half* each vertical angle of these pyramids, and *double* each solid angle at their bases.

The angles made by one plane with another, must be ascertained, either by measurement or by computation, according to circumstances. But, the general theory being thus explained and illustrated, the further application of it is left to the skill and ingenuity of geometers; the following simple examples, merely, being added here.

Ex. Let the solid angle at the vertex of a square pyramid be bisected.

1st. Let a plane be drawn through the vertex and any two opposite angles of the base, that plane will bisect the solid angle at the vertex; forming two trilateral angles, each equal to half the original quadrilateral angle.

2dly. Bisect either diagonal of the base, and draw *any* plane to pass through the point of bisection and the vertex of the pyramid; such plane, if it do *not* coincide with the former, will divide the quadrilateral solid angle into two equal quadrilateral solid angles. For this plane, produced, will bisect the great circle diagonal of the spherical parallelogram cut off by the base of the pyramid; and any great circle bisecting such diagonal is known to bisect the spherical parallelogram, or square; the plane, therefore, bisects the solid angle.

Cor. Hence an indefinite number of planes may be drawn, each to bisect a given quadrilateral solid angle.

Ex. 2. Determine the solid angles of a regular pyramid with hexagonal base, the altitude of the pyramid being to each side of the base, as 2 to 1.

Ans. Plane angle between each two lateral faces .... 125° 22′ 35″
———— between the base and each face .... 66° 35′ 12″.
Solid angle at the vertex 89·60648 } The max. angle
Each ditto at the base .. 218·19367 } being 1000.

CHAPTER VII.

Logarithmic Computation of Spherical Triangles.

1. FOR the purposes of exemplifying the rules and formulæ in the preceding chapter, and of assisting the student in deducing the logarithmic computations from the analytical expressions, a few problems are here added.

Example I.

In the right angled spherical triangle ABC, right angled at A, given the hypothenuse $a = 64° 40'$, and one leg $b = 42° 12'$. Required the rest.

This example falls under case 2, section 2, of the preceding chapter, where

$$\sin \text{B, a req. angle} = \frac{\text{sin given side}}{\text{sin hypoth.}}.$$

$$\cos c\text{, the side req.} = \frac{\text{cos hypoth.}}{\text{cos given side}}.$$

$$\cos \text{C, a req. angle} = \text{tan given side} \times \text{cot hypoth.}$$

Hence the following logarithmic operation.

From log sin 42° 12′.. 9·8271887	From log cos 64° 40′..9·6313258
Take log sin 64° 40′:.9·9560886	Take log cos 42° 12′..9·8697037
Rem. sin B. 48° 0′..9·8711001	Rem. cos *c*..54° 43′..9·7616221

To log tan 42° 12′ .... 9·9574850
Add log cot 64° 40′ .... 9·6752372

The sum is log cos C 64° 35′ .... 9·6327222

Here 10 are added to the index of the *remainder*, and taken from the index of the *sum;* conformably with note 2, art. 31 of the preceding chapter.

2dly. *To compute the same by means of Napier's circular parts.*

Here, if the leg b be assumed as the middle term, $(90° - a)$, and $(90° - \text{B})$ are the opposite parts.

and sin mid. = rect cos op. becomes,

$$\sin b = \sin a \sin \text{B}.$$

$$\text{Hence } \sin \text{B} = \frac{\sin b}{\sin a} = \frac{\text{sin given leg}}{\text{sin hypoth.}};$$

this agrees with the foregoing process.

Again, if $(90° - \text{C})$ be the middle part, then $(90° - a)$ and b are adjacent parts, and

sin mid. = rect tan adja. becomes

cos C = cot a tan b = cot hypoth. × tan given side; this also agrees with the preceding.

3dly. If $(90° - a)$ be made the middle part, then b and c are the opposite parts, and

sin mid. = rect cos oppos. becomes

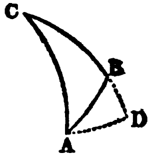
$$\cos a = \cos b \cos c;\ \text{whence } \cos c = \frac{\cos a}{\cos b},$$

that is, $\cos c = \frac{\text{cos hypoth.}}{\text{cos given leg}}$; which also agrees with the preceding. Hence the logarithmic computation need not be repeated.

Example II.

Given in a spherical triangle, the quadrantal side CA = 90°, an adjacent angle C = 42° 12′, and the opposite angle B = 115° 20′; to find the other angle and sides.

Suppose the side CB produced until CD = CA = 90°: then, it is evident from chap. vi. art. 33, that both the angles, CAD, and D, are = 90°, and consequently that AD is the measure of the angle C, and therefore = 42° 12′. It is also evident that ABD, as well as ACD, is a right angled triangle, and that ∠ ABD = 180° − ABC = 64° 40′. Hence, to find AB, &c. we make use of the triangle ABD, of which we determine the hypothenuse and oblique angles by case 4, of right angled triangles. Thus,



$\text{sin hypoth.} = \frac{\text{sin given side}}{\text{sin op. angle}}$.	$\text{sin req. an.} = \frac{\text{cos given angle}}{\text{cos given side}}$.
From log sin 42° 12′..9·8271887	From log cos 64° 40′..9·6313258
Take log sin 64° 40′..9·9560886	Take log cos 42° 12′..9·8697037
Rem. sin AB .. 48° ..9·8711001	Rem. sin BAD 35°17′ or cos CAB 54°43′ } 9·7616221

sin side req. = tan given side × cot op. angle.

To log tan 42° 12′....9·9574850
Add log cot 64° 40′....9·6752372

The sum is sin BD 25° 25′ or cos BC 64° 35′ ....9·6327222

Example III.

In a right angled spherical triangle given the hypothenuse = 64° 40′, and an adjacent angle = 64° 35′; to find the rest.

Example IV.

Given one leg = 42° 12′, and its opposite angle = 48°; to find the rest.

Example V.

Given a leg = 54° 43′, and its adjacent angle = 48°; to find the rest.

Example VI.

Given the two legs = 54° 13′; and 42° 12′, respectively; to find the rest.

Example VII.

Given the two oblique angles = 48° and 64° 35′ respectively; to find the rest.

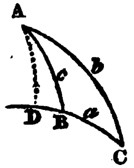
Example VIII.

Given a quadrantal side, one of the other sides = 115° 9′, and the angle comprehended between them = 115° 55′; to find the rest.

Ans. Angles 101° 4′ and 117° 34′, side 113° 18′.

Example IX.

Given in an oblique angled spherical triangle, the side a = 44° 13′ 45″, b = 84° 14′ 29″, and their included angle C = 36° 45′ 28″; to find the rest.



This example corresponds with case 2, prob. 2, of oblique angled spherical triangles; and may first be solved by means of the subsidiary arc, in the manner there explained.

Thus, first find φ, so that

$$\tan \varphi = \cos \mathrm{C} \tan b.$$

To log cos C = cos 36° 45′ 28″ .... 9·9037261
Add tan b = tan 84° 14′ 29″ .... 10·9963395

tan φ = tan 82° 49′ 33″ .... 10·9000656

This arc φ exceeds a, therefore the perpendicular AD from the vertical angle falls on the base produced: hence the 2d expression becomes

$$\cos c = \frac{\cos b \cos (a \backsim \phi)}{\cos b}.$$

Hence, to log cos 84° 14′ 29″ .... 9·0014632
add cos 38° 35′ 48″ .... 9·8929604

from the sum .... 18·8944236
take cos 82° 49′ 33″ .... 9·0965132

Rem. cos c = cos 51° 6′ 11″ .... 9·7979104

To find the remaining parts use the known proportion of the sines of sides to the sines of their opposite angles; thus

As sin c .... 51° 6′ 11″ .... 9·8911340
To sin C .... 36° 45′ 28″ .... 9·7770158
So is sin a...... 44° 13′ 45″ .... 9·8435629
To sin A .... 32° 26′ 7″ .... 9·7294447
And so is sin b .. 84° 14′ 29″ .... 9·9978028
To sin B .... 130° 5′ 21″ .... 9·8836846

Here the logarithmic sine 9·8836846 answers either to 49° 54′ 39″ or to its supplement 130° 5′ 21″; the former of which is the exterior angle ABD, the latter the angle B of the triangle.

2d *Method, by Napier's Analogies.*

Taking the 14th and 15th formulæ at the end of sect. 4, of the preceding chapter, we have

$$\tan \tfrac{1}{2}(\mathrm{B} - \mathrm{A}) = \cot \tfrac{1}{2}\mathrm{C}\, \frac{\sin \frac{1}{2}(b - a)}{\sin \frac{1}{2}(b + a)};$$

$$\text{and } \tan \tfrac{1}{2}(\mathrm{B} + \mathrm{A}) = \cot \tfrac{1}{2}\mathrm{C}\, \frac{\cos \frac{1}{2}(b - a)}{\cos \frac{1}{2}(b + a)}.$$

The log. computation will therefore stand thus:

To log cot $\frac{1}{2}$C	18° 22′ 44″....	10·4785395
Add log sin $\frac{1}{2}(b-a)$	20° 0′ 22″....	9·5341789
From the sum		20·0127184
Take log sin $\frac{1}{2}(b+a)$	64° 14′ 7″....	9·9545255
Rem. log tan $\frac{1}{2}$(B−A)	48° 49′ 38″....	10·0581929
Also, to log cot $\frac{1}{2}$C ..	18° 22′ 44″....	10·4785395
Add log cos $\frac{1}{2}(b-a)$	20° 0′ 22″....	9·9729690
From the sum		20·4515085
Take log cos $\frac{1}{2}(b+a)$	64° 14′ 7″....	9·6381663
Rem. log tan $\frac{1}{2}$(B+A)	81° 15′ 44″....	10·8133422

Hence 81° 15′ 44″ + 48° 49′ 38″ = 130° 5′ 22″ = B,
and 81° 15′ 44″ − 48° 49′ 38″ = 32° 26′ 6″ = A;
agreeing nearly with the result of the former computation.

Then to find c, use the proportion, as sin A : sin a :: sin C : sin c = sin 51° 6′ 12″.

Here it would seem, from a comparison of the methods, that the first is rather quickest in operation, while the last is probably the easiest to remember, and provides best against the occasions of ambiguity.

Example X.

In an oblique angled spherical triangle ABC, given the side $c = 114^\circ\, 30'$, side $a = 56^\circ\, 40'$, and the angle C opposite the first side $= 125^\circ\, 20'$: to find the rest.

Ans. $A = 48^\circ\, 30'$, $B = 62^\circ\, 54'$, $b = 83^\circ\, 12'$.

Example XI.

Given $A = 48^\circ\, 30'$, $C = 125^\circ\, 20'$, $c = 114^\circ\, 30'$; to find the rest.

Example XII.

Given $a = 56^\circ\, 40'$, $c = 114^\circ\, 30'$, $B = 62^\circ\, 54$; to find the rest.

Example XIII.

Given $A = 48^\circ\, 30'$, $C = 125^\circ\, 20'$, $b = 83^\circ\, 12'$; to find the rest.

Example XIV.

Given $a = 56^\circ\, 40$, $b = 83^\circ\, 12'$, $c = 114^\circ\, 30'$; to find the rest.

Example XV.

Given $A = 48^\circ\, 30'$, $B = 62^\circ\, 54'$, $C = 125^\circ\, 20'$; to find the rest.

⁂ For more examples see chap. x.

CHAPTER VIII.

On Projections of the Sphere.

SECTION I.

Astronomical Definitions.

1. SINCE the figure of the earth differs but little from that of a sphere, it is usual in the greater part of the

inquiries and computations of astronomers, to proceed as though it were a sphere in reality; and since, to an observer on the earth, the heavens appear as a very large concave sphere, every part of which is equidistant from him, it has been found expedient to imagine various lines and circles to be described upon the earth, and the planes of several of them to be extended every way until they mark other similar lines and circles upon the imaginary concave sphere of the heavens. Some of these it now becomes necessary to explain.

2. The *axis* of the earth is an imaginary right line passing through the centre, about which line it is supposed to turn uniformly once in a natural day.

3. The extremities of this axis are called the *poles* of the earth.

4. That great circle of the earth, the poles of which are the poles of the earth, is called the *equator*.

5. If the axis of the earth be supposed produced both ways to the concave heavens, it is then called the *axis of the heavens;* its extremities are called the *poles of the heavens;* and the circumference formed by extending the plane of the equator to the celestial concavity is called the *celestial equator*, or the *equinoctial.*

6. A secondary to the equator drawn through any place on the earth, and passing through the poles, is called the *meridian* of that place.

7. The *latitude* of any place upon the surface of the earth, is its distance from the equator measured on an arc of the meridian passing through it. A less circle passing through any place parallel to the equator is called a *parallel of latitude.* Places that lie between the equator and the north pole have *north latitude;* if they lie between the equator and the south pole they have *south latitude.*

8. All places that lie under the same meridian have the same *longitude;* and those places which lie under different meridians have different longitudes. The *difference of longitude* between any two places, is the dis-

tance of their meridians measured in degrees, &c. upon the equator.

9. The *sensible horizon* is a circle, the plane of which is supposed to touch the spherical surface of the earth, in the place of the spectator whose horizon it is. The *rational horizon* is a circle whose plane passes through the centre of the earth, parallel to the plane of the sensible horizon. The radius of the earth being exceedingly minute compared with that of the celestial sphere, the sensible and rational horizon may, in many astronomical inquiries, be supposed, without error, to coincide.

10. Great circles which are drawn as secondaries to the rational horizon, are called *vertical circles;* they serve to measure the altitude or the depression of any celestial object.

11. The two points in which all the vertical circles that can be drawn to any rational horizon meet, are called, the one above the spectator the *zenith,* and that which is below him the *nadir.*

12. *Almucantars*, or *parallels of altitude*, are circles parallel to the horizon, or whose poles are the zenith and nadir. All the points of any one almucantar are at equal altitudes above the horizon.

13. The real motion of the earth about the sun once in a year, gives rise to an apparent motion of the sun about the earth in the same interval of time. The circle in which the sun appears to move is called the *ecliptic;* the angle in which it crosses the equinoctial the *obliquity of the ecliptic;** and the two points where it intersects that circle, the *equinoxes.*

---

* The obliquity of the ecliptic is a *variable* quantity, oscillating between certain limits which it never passes. According to the profound investigations of *Laplace* in physical astronomy, the obliquity may always be determined very nearly by this formula, viz.

$$\left.\begin{array}{l} 23^\circ\,28'\,23''{\cdot}05 - 1191''{\cdot}2184\,[1 - \cos(t\,18''{\cdot}94645)] \\ \qquad - 3347''{\cdot}0496\,\sin(t\,32''{\cdot}11575) \end{array}\right\}$$

where t denotes the number of years run over from 1750; it is

14. The distance of the sun, or of any of the heavenly bodies, from the equator, measured on an arc of the meridian, is called the *declination; north* or *south*, according as the body is situated north or south of the equator.

15. Secondaries to the celestial equator are called *circles of declination;* of these, twenty-four, which divide the equator into equal parts of 15° each, are called *hour circles;* because the sun in his apparent diurnal motion passing over 360° of a circle parallel to the equator, goes through $\frac{1}{24}$th of them, or 15°, in an hour.

16. *The right ascension* of a celestial body is an arc of the equinoctial, intercepted between one of the equinoxes, and a declination circle passing through the body; it is measured according to the order of the sun's apparent motion through the twelve signs.

17. The *longitude* of a heavenly body is an arc of the ecliptic, contained between the 1st point of Aries (that is, one of the equinoctial points), and a secondary to the ecliptic, or a circle of latitude passing through the body.

18. The *latitude* of a body is its distance from the ecliptic measured upon a secondary to that circle. And the angle formed at the body by two great circles, one passing through the pole of the equator, the other through the pole of the ecliptic, is called the *angle of position.*

19. The *tropics* are two circles parallel to the equinoctial, and touching the ecliptic at the two points where it is most remote from the equator; that is to say, the first points of Cancer and of Capricorn; the former is denominated the *tropic of Cancer*, the latter the *tropic of Capricorn.*

negative *before*, positive *after*, that epoch. This theorem is found to answer very well up to the time of *Pytheas*, 350 years before the Christian era. The obliquity at the beginning of 1816 is 23° 27′ 49″·2.

20. The points where the tropics touch the ecliptic are called *solstices*, because the sun when in either of them appears to be at a stand with regard to his declination.

21. *Colures* are two secondaries to the equinoctial: one, passing through the equinoctial points, is called the *equinoctial colure;* the other, passing through the solstices, is called the *solstitial colure.*

22. Small circles drawn at the distance of 23° 28′ (or correctly 23° 27′ 49″) from the north and south poles of the equator, are called *polar circles;* the former the *arctic*, the latter the *antarctic* circle.

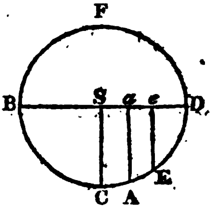
23. That vertical circle which intersects the meridian of any place at right angles, is called the *prime vertical:* the points where it cuts the horizon are the *east* and *west* points; at the distance of 90° from each of these on the horizon are the *north* and *south;* all four being called *cardinal* points.

24. The distance on the horizon of a vertical circle that passes through any body from the north or south points is the *azimuth* of that body; the distance of the same circle from the east or west points is the *amplitude.*

25. In order to represent on a plane the celestial sphere with all its circles great and small, the ancients invented two kinds of *projection.* The first, named by them *Analemma*, has since received the name of *Orthographic Projection.* The second, originally denoted by the generic term *Planisphere*, received from the Jesuit *Aguilon* the name of *Stereographic Projection.* The adjective *orthographic* is given to the former, because it is produced by lines which fall at *right angles* upon the plane which represents the sphere. That of *stereographic* was given to the other, because it results from the intersection of two *solids*, a sphere and a cone.

SECTION II.

Orthographic Projection.



26. In this projection the eye is supposed at an infinite distance: in which case a great circle BCDF is apparently reduced to a right line equal to its diameter BD, the eye being imagined indefinitely distant in the direction SC. Then, also, every arc CA which has its origin at the apparent centre, has for its projection a right line S*a* equal to the *sine* of that arc. The quadrant CB or CD will be projected into *its* sine, or radius. An arc as AE = CE − CA, will be projected into *ae*, = sin CE − sin CA = (by equa. U, chap. iv.) $2 \sin \frac{1}{2}$ (CE − CA) $\cos \frac{1}{2}$ (CE + CA) = $2 \sin \frac{1}{2}$ AE cos (CA + $\frac{1}{2}$AE) .... (1.)

27. Every arc, as DA, from the edge of the disc towards the centre, is projected into its *versed sine*. D*a*. The quadrant DC, therefore, is projected into the radius DS the versed sine of 90°; and the semicircle DCB into the diameter DB, the versed sine of 180°.

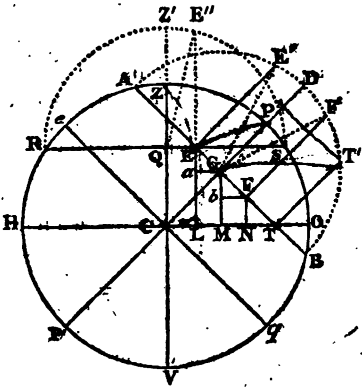
28. What is here remarked of the circle BCDF applies equally to all great circles which intersect at S and form the visible hemisphere: each of these semicircles is reduced to its diameter, and the hemisphere is reduced to a disc.

29. In this projection every circle, great or small, whose plane prolonged does not pass through the eye, will be seen obliquely, and under an *elliptical* form: for an oblique circle making throughout the same angle with the plane of projection, its several parallel ordinates are all reduced in a constant ratio; therefore, the projected ordinates are all in a constant ratio to the corresponding ordinates of the circle of equal diameter

on the plane of projection, and together constitute an ellipse. (*Hutton on Ellipse*, prop. 3, cor. 1).

30. In general, in the orthographic projection every great circle perpendicular to the plane of projection is represented by its diameter; and every circle perpendicular to that plane is represented by a chord of the primitive circle equal to its diameter.

Thus if, by way of showing the use of this projection, we assume the meridian for the plane of projection, the horizon will be represented by its diameter HO: the prime vertical by its diameter ZV, which cuts the former perpendicularly: the six o'clock hour circle will be represented by its diameter which is the axis PP′, making with the horizon the angle PCO = the height of the pole = the latitude = L: the equator will be projected into its diameter *eq*, making with the horizon an angle HC*e* = 90° − L: the parallels to the equator will be represented by chords, such as AB parallel to the diameter of the equator: the almucantars are projected into chords, such as RS, parallel to the horizontal diameter HO.



31. AB being the projection of a certain parallel, suppose that the star which in its apparent motion describes that parallel, has its inferior transit of the meridian at B. The point B which is its place on the sphere, is also its place *then* in the projection. The star being in the horizon at T, OT will be the versed sine of its azimuth, and CT the sine of its amplitude. Also, refer-

ring the point T to the chord BA, BT will be the versed sine of the arc described from the inferior transit of the star to its rising, or the *semi-nocturnal arc*; TG will be the sine of the arc which remains to be described before it reaches the 6 o'clock hour circle, or TG will be the sine of the arc which deducted from 90° will leave the semi-nocturnal arc; or TG will be the sine of the arc which added to 90°, equivalent to AG, will give the *semi-diurnal arc* represented by AT.

On AB as a diameter describe the semi-circle ADB; from any points F and E answering to the position of the star at different instants, erect the perpendiculars FF′, EE′; produce CP to D; then it is evident that if the semi-circle ADB were elevated perpendicularly on the plane of the meridian, F, G, and E, would be the respective projections of the points F′, D, and E′.

32. To find the value of the arc DE′ projected into the rectilinear portion GE, we have

$$\text{rad} : \sin \text{DE}' :: \text{AG} : \text{GE}; \text{ whence}$$

$$\sin \text{DE}' = \frac{\text{GE} \times \text{rad}}{\text{AG}} = \frac{\text{GE}}{\text{GA}} = \frac{\text{GE}}{\sin \text{PA}} = \frac{\text{GE}}{\cos \text{AC}} = \frac{\text{GE}}{\cos \text{D}} \ldots (2.)$$

Hence, to know the arc which answers to GE, GF, GT, &c. we must divide each of those lines estimated from the middle of the chord by the sine of the chord's distance from its pole.

Again, from the triangle CGT we have

$$\text{GT} = \text{CG} \tan \text{GCT} = \sin \text{D} \tan \text{L} \ldots\ldots (3.)$$

D being the declination of the parallel;

$$\therefore \frac{\text{GT}}{\cos \text{D}} = \frac{\sin \text{D}}{\cos \text{D}} \tan \text{L} = \tan \text{D} \tan \text{L} = \sin \text{DT}' \ldots\ldots (4.)$$

From the same triangle we have, also,

$$\text{CT} = \sin \text{ampl.} = \cos \text{azim.} = \frac{\text{CG}}{\cos \text{GCT}} = \frac{\sin \text{D}}{\cos \text{L}} \ldots\ldots (5.)$$

33. Through the point E, a projected place of a star, draw the chord RS parallel to the horizon, QE will be to the radius QR the cosine of the azimuth, and EL perpendicular to the horizon will be the sine of the alti-

tude. To determine this from the projection, draw $\mathrm{G}a$ parallel to the horizon, then

$$\begin{aligned}\sin \mathrm{A} &= \mathrm{EL} = \mathrm{L}a + a\mathrm{E} = \mathrm{GM} + a\mathrm{E} \\ &= \mathrm{CG} \sin \mathrm{GCM} + \mathrm{EG} \sin \mathrm{EG}a \\ &= \sin \mathrm{D} \sin \mathrm{L} + \mathrm{AG} \sin \mathrm{DE'} \cos \mathrm{GCO} \\ &= \sin \mathrm{D} \sin \mathrm{L} + \cos \mathrm{D} \cos \mathrm{L} \cos \text{hour angle} \\ &= \sin \mathrm{D} \sin \mathrm{L} + \cos \mathrm{D} \cos \mathrm{L} \cos \mathrm{H} \ldots\ldots (6.)\end{aligned}$$

Put $\Delta = 90° - \mathrm{D}$, $\mathrm{E} = 90° - \mathrm{L}$, and $z = 90° - \mathrm{A}$; then will the last equation become

$$\cos z = \cos \Delta \cos \mathrm{E} + \sin \Delta \sin \mathrm{E} \cos \mathrm{H},$$

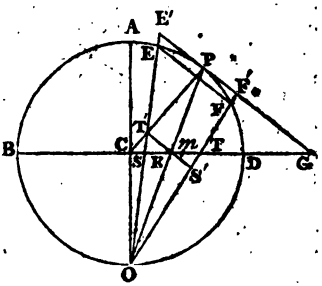
which accords precisely with the fundamental equation (2) of spherical trigonometry; and the equa. (1 and 4) may readily be deduced from the same diagram.

SECTION III.

Stereographic Projection.

34. In this projection, which appears to have been invented by Hipparchus, all circles, whether great or small, are represented by *circles;* and a second property, equally general and more curious, is that in the projection all the circles make respectively *the same angles as on the sphere.*

35. Let ABOD be a great circle of the sphere, O the place of the eye: the diameter OCA being drawn, and the diameter BCD perpendicular to it, BCD will be the orthographic projection of a great circle perpendicular to the visual ray OA, or of the circle one of whose poles is O: it is on the plane of this circle that it is proposed to describe all the circles of the sphere, as they would appear at the



point O. The circle, then, whose diameter is BD is the plane of projection; the point C its centre; A and O its poles; and the point C is evidently the projection of the point A.

36. Let P be any point assumed on the circumference OBAD: take PE = PF, and draw the chord EF, it will be the orthographic projection, or the diameter of a small circle whose pole is P. Draw EO, FO, to cut BD in S and T, ST will be the projection of the chord EF on BD; and we propose to demonstrate that ST is the diameter of a *circle* which will be the projection of the circle described on EF. Now it is evident that rays from all points of the circumference of the circle whose diameter is EF to meet at O will form the surface of an oblique cone whose vertex will be O, and circle about EF its base; of which all sections parallel to that base will (*Hutton's Geometry*, theor. 113) be circles. In order to determine the section of this cone whose orthographic projection is ST, we may proceed thus:

$$\text{meas. of } \angle FEO \text{ is } \tfrac{1}{2}EO = \tfrac{1}{2}OD + \tfrac{1}{2}DF = 45^\circ + \tfrac{1}{2}DF;$$
$$\text{meas. of } \angle STO \text{ is } \tfrac{1}{2}OB + \tfrac{1}{2}DF = 45^\circ + \tfrac{1}{2}DF:$$

therefore $FEO = STO$; and consequently,

$$EFO = 180^\circ - FEO - FOE = 180^\circ - STO - TOS = OST.$$

The triangles EFO, TSO, then, are similar; yet the lines EF, ST, are not parallel, but are what is technically denominated *anti-parallel*, or *sub-contrary*. Suppose, however, the cone EOF to be turned half round upon the axis PO, then (since both slant sides OE, OF, make equal angles with OP) OT would become OT′, and OS would become OS′; in that case T′S′ would be parallel to the original chord EF, and the section of the cone (which *can* in no respect of magnitude or shape differ from the section projected into ST) would evidently be a circle. ST is, therefore, the diameter of a circular section. Thus *every circle, whether oblique or not to the visual ray directed to its pole, will be represented on the projection by a circle.*

37. Draw from the centre to the pole P, the radius CP, and through the point P the tangent KPFG, to meet in G the plane of projection BD prolonged. Then,

$$\text{meas. of} \angle \mathrm{GPK} = \tfrac{1}{2}\mathrm{PDO} = 45^\circ + \tfrac{1}{2}\mathrm{PD};$$

$$\text{meas. of} \angle \mathrm{PKG} = \tfrac{1}{2}\mathrm{BO} + \tfrac{1}{2}\mathrm{PD} = 45^\circ + \tfrac{1}{2}\mathrm{PD};$$

$$\text{therefore } \mathrm{GPK} = \mathrm{GKP}, \text{ and } \mathrm{PG} = \mathrm{GK}:$$

that is, *the tangent* PG *is projected into a line* KG *equal to it.*

38. Since POE and POF are equal, the line OP will not bisect ST; but KS < KT. Bisect ST in *m*, then *m*S = *m*T = radius of circle of projection; and CS, C*m*, CT, will be in arithmetical progression. Hence,

$$\mathrm{C}m = \tfrac{1}{2}\mathrm{CT} + \tfrac{1}{2}\mathrm{CS} = \tfrac{1}{2}\tan\tfrac{1}{2}\mathrm{AF} + \tfrac{1}{2}\tan\tfrac{1}{2}\mathrm{AE} =$$

$$\frac{\sin\frac{1}{2}(\mathrm{AF}+\mathrm{AE})}{2\cos\frac{1}{2}\mathrm{AF}\cos\frac{1}{2}\mathrm{AE}} = \frac{\sin \mathrm{AP}}{2\cos\frac{1}{2}(\mathrm{AP}+\mathrm{PE})\cos\frac{1}{2}(\mathrm{AP}-\mathrm{PE})}.$$

Let d = C*m* distance of the centre *m* from the centre C of the plane of projection, Δ = PE = polar distance of the circle EF, D = AP = distance of the two poles, $r = m\mathrm{S} = m\mathrm{T}$: then

$$d = \frac{\sin \mathrm{AP}}{\cos \mathrm{AP} + \cos \mathrm{PE}} = \frac{\sin \mathrm{D}}{\cos \mathrm{D} + \cos\Delta} \ldots\ldots (7.)$$

$$r = \tfrac{1}{2}\mathrm{ST} = \tfrac{1}{2}(\mathrm{CT} - \mathrm{CS}) = \tfrac{1}{2}(\tan\tfrac{1}{2}\mathrm{AF} - \tan\tfrac{1}{2}\mathrm{AE})$$

$$= \frac{\sin\frac{1}{2}(\mathrm{AF}-\mathrm{AE})}{2\cos\frac{1}{2}\mathrm{AF}\cos\frac{1}{2}\mathrm{AE}} = \frac{\sin\Delta}{\cos \mathrm{D} + \cos\Delta} \ldots\ldots (8.)$$

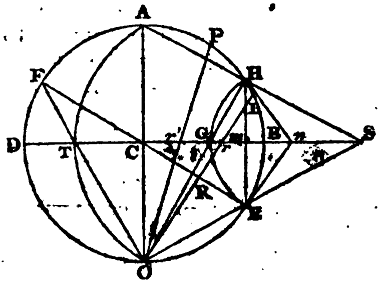
[See formulæ (U), &c. chap. iv.]

Consequently, $d : r :: \sin \mathrm{D} : \sin\Delta \ldots\ldots (9.)$

From these three theorems the whole doctrine of stereographic projection may be deduced, by tracing the mutations of D and Δ. The chief maxims and principles of construction may also be developed geometrically, thus:

39. Beginning with great circles, let PE = 90°, and PF = PE; then will EF be a diameter. Draw the right lines OTF, OES, bisect TS in *r*; then *r*S = *r*T, will be the radius of the circle into which the great circle whose diameter is EF will be projected. Through O and *r*, draw OR*r*I. The circle described on ST will pass

through O, because KOE is a right angle. In like manner, it will pass through A, because CS is perpendicular to the middle of AO. Therefore $r\mathrm{O} = r\mathrm{S} = r\mathrm{T}$; conseq. $\mathrm{SO}r = \mathrm{OS}r$, and $\mathrm{C}r\mathrm{O} = 2\,\mathrm{OS}r$. $\mathrm{CO}r = 90^\circ - \mathrm{C}r\mathrm{O} = 90^\circ - 2\,\mathrm{OS}r = 90^\circ - (\mathrm{DO} - \mathrm{BE}) = 90^\circ - 90^\circ + \mathrm{BE} = \mathrm{BE}$. That is, $\mathrm{CO}r$ = *inclination of the plane of the great circle to the plane of projection.*



Hence, $\mathrm{O}r$ is manifestly = sec. inclination, and $\mathrm{C}r$ = tan inclination, to rad CO.

Thus, with the radius = sec D, and at the distance from the centre = tan D, it will be easy to describe the circle.

Again, take $\mathrm{AI} = 2\mathrm{BE} = 2\mathrm{AP} = 2\mathrm{D}$, and draw OI, the point of intersection r with DS will be the centre, and $r\mathrm{O}$ the radius.

Also, since meas. of $\mathrm{CO}r = \mathrm{BE}$, meas. of $\mathrm{OCR} = \mathrm{OE} = 90^\circ - \mathrm{BE}$; we have $\mathrm{COR} + \mathrm{OCR} = 90^\circ$, and conseq. $\mathrm{CRO} = 90^\circ$. Hence, by drawing $\mathrm{OR}r$ perpendicular to FE, we find the centre r and the radius $r\mathrm{O}$: also $\mathrm{CR} = \sin \mathrm{BE} = \sin \mathrm{D}$, $\mathrm{OR} = \cos \mathrm{D}$.

40. $\mathrm{O}r$, therefore, will always make with OC an equal equal to the inclination or distance of the two poles. Let there be, then, a second circle whose inclination shall be $\mathrm{CO}r'$; the radius of its projection will be $r'\mathrm{O}$, and the radii $r\mathrm{O}$, $r'\mathrm{O}$, will make at their points of intersection an angle $r\mathrm{O}r'$ which will be their difference of inclination, or the mutual inclination of the two circles. This is a particular case of the general theorem.

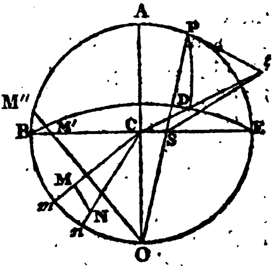
41. Suppose now that $\mathrm{AP} = 90^\circ$, P will coincide with B, and the pole of the circle will be upon the limit of

the projection. Let $BH = BE = \Delta$ polar distance of the circle to be projected, the chord HE will be perpendicular to DB. Draw OH and OES, GS will be the diameter sought. Bisect GS in n, n will be the centre, $nE = nH = nS = nG$, will be the radius of the projected circle.

Meas. of $EnG = 2nSE = OD - BE = 90° - BE = 90° - BCE$
$CnE + nCE = 90°$ $\therefore$ $CEn = 90°$ $\therefore$ $En = \tan BE = \tan \Delta$, and $Cn = \sec BE = \sec \Delta$.

These values serve for all circles which have their pole on the circumference of the circle of projection. If these are great circles, then $\Delta = 90°$, and $\tan \Delta$, $\sec \Delta$ are infinite: consequently, the centres of the projections falling at an infinite distance from C, the projections themselves will be *right lines* passing through C, and intersecting under the angles which such circles, taken two and two, form on the sphere.

42. Let O be the point of observation, or place of the eye, A the pole of the projection, BDECB the plane of projection, PD an arc of a great circle which has its origin at any point whatever of the circle OBPE, Pt the tangent of PD; Ct will be the secant, and S the projection of P. Draw St: then from the rectilineal triangle SCt we shall have,



$$
\begin{aligned}
St^2 &= CS^2 + Ct^2 - 2CS \, . \, Ct \, . \cos SCt \\
&= \tan^2 \tfrac{1}{2}AP + \sec^2 PD - 2 \tan \tfrac{1}{2}AP \sec PD \cos DE. \\
&= \tan^2 \tfrac{1}{2}AP + \tan^2 PD + 1 - 2 \tan \tfrac{1}{2}AP \sec PD \cos DE \\
&= \sec^2 \tfrac{1}{2}AP + \tan^2 PD - 2 \tan \tfrac{1}{2}AP \sec PD \cos DE.
\end{aligned}
$$

But the spherical triangle PDE right angled in E gives, (chap. vi. equa. 6),

$$\cos PD = \cos PE \cos DE = \sin AP \cos DE;$$

and, therefore, $\sec PD = \dfrac{1}{\cos PD} = \dfrac{1}{\sin AP \cos DE}$.

Substituting this in the last value of St^2, we have

$$st^2 = \sec^2 \tfrac{1}{2}\mathrm{AP} + \tan^2 \mathrm{PD} - \frac{2 \tan \frac{1}{2}\mathrm{AP} \cos \mathrm{DE}}{\sin \mathrm{AP} \cos \mathrm{DE}}$$

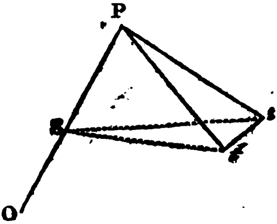
$$= \sec^2 \tfrac{1}{2}\mathrm{AP} + \tan^2 \mathrm{PD} - \frac{2 \tan \frac{1}{2}\mathrm{AP}}{2 \sin \frac{1}{2}\mathrm{AP} \cos \frac{1}{2}\mathrm{AP}}$$

$$= \sec^2 \tfrac{1}{2}\mathrm{AP} + \tan^2 \mathrm{PD} - \sec^2 \tfrac{1}{2}\mathrm{AP} = \tan^2 \mathrm{PD}.$$

Therefore $st = \tan \mathrm{PD} = \mathrm{P}t$.

Consequently, *the tangent* Pt *of an arc of a great circle terminated at the plane of projection, is projected into a right line equal to it.*

Let Pt, and Pt', two such tangents, be connected by the right line tt' which will be in the plane of projection. Let st, st', be the projections of those tangents; the triangles $t\mathrm{P}t'$, tst', are (from the above) equal in all respects: therefore the angles opposite to the common side tt' will be the same in both: consequently, *the tangents of any two arcs terminated at the plane of projection, are projected into lines which are respectively equal to them, and which form an equal angle.* Hence, two circles which intersect in P on the sphere, form on the projection an angle equal to that which they make on the sphere; because, at the point of intersection the elements of the arcs coincide with those of their tangents. Therefore, *all great circles intersect mutually on the plane of projection under the same angle as on the sphere; so also do little circles which intersect at the same points, and have, by consequence, common tangents.*

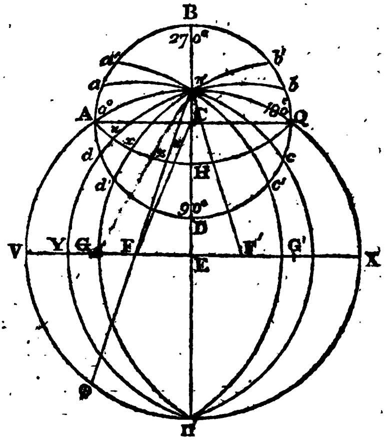


43. By way of showing the application of these principles, let us suppose that the eye is at the south pole of the equator. The plane of projection will then be the equator itself; the centre of the equator will represent the north pole; AP (fig. to art. 35) will be $= 0$; the projections of the parallels to the equator will all

have for a common centre that of the projection; and the radii of those circles will be the tangents of the *halves* of the polar distances. Thus,

for the polar circle .... $r = \tan \frac{1}{2}(23°28') = \tan 11°44'$
tropic of cancer . $r = \tan \frac{1}{2}(66°32') = \tan 33°16'$
tropic of capricorn $r = \tan \frac{1}{2}(113°28') = \tan 56°44'$
for the antarctic circle $r = \tan \frac{1}{2}(156°32') = \tan 78°16'$
for any latitude L .... $r = \tan \frac{1}{2}(90° - L) = \tan 45° - \frac{1}{2}L$
or, if the lat. be south $r = \tan(45° + \frac{1}{2}L)$.

As for the meridians, whose planes all pass through the place of the eye, they all become diameters which divide the equator in its several degrees, and form at the centre of the projection angles equal to the differences of longitude. For these circles $d = \infty$, and $r = \infty$ (art. 38).



This kind of projection, the easiest of all to describe, serves very conveniently for eclipses of the sun. The meridians and the parallels are herein divided mutually into degrees: those of the parallels are equal; those of the meridians unequal; for the expression for one of their degrees is,

$$\delta = \tan \tfrac{1}{2}(\Delta + 1°) - \tan \tfrac{1}{2}\Delta = \frac{\sin 30'}{\cos \frac{1}{2}\Delta \cos \frac{1}{2}(\Delta + 1°)}$$

$$= \frac{\sin 30'}{\cos \frac{1}{2}\Delta (\cos \frac{1}{2}\Delta \cos 30' - \sin \frac{1}{2}\Delta \sin 30')}$$

$$= \frac{\tan 30'}{\cos \frac{1}{2}\Delta - \tan 30' \sin \frac{1}{2}\Delta \cos \frac{1}{2}\Delta} = \frac{\tan 30' \sec^2 \frac{1}{2}\Delta}{1 - \tan 30' \tan \frac{1}{2}\Delta}.$$

44. On a planisphere of this kind, the stars are placed according to their right ascensions and declinations. Then the ecliptic is traced, as well as its poles, the circles of latitude all intersecting mutually at those two poles, and then the parallels to the ecliptic.

Thus, on the radius marked 270°, at a distance Cπ from the centre = tan 11° 44′, we mark the north pole π of the ecliptic. From that point, with an opening of the compasses = cosec 23° 28′ we mark the point E, or, which amounts to the same, we make CE = cot 23° 28′. Through the point E we draw the indefinite perpendicular VEX, which is evidently the locus of the centres of all the circles of latitude intersecting mutually in Π and π: E will be the centre of that circle of latitude which passes through 0° and 180°, or of the equinoctial colure, which will be the circle VπXΠ. πEΠ will be the solstitial colure.

In general, make EπG′ = longitude: from the centre G′ with the radius G′π we describe a circle, it will be the circle of latitude which answers to the longitude supposed. Repeating the same operation on the other side of the line πE, we shall have the circles of latitude of the other hemisphere. For the circles parallel to the ecliptic we employ the formulæ

$$d = \frac{\tan\frac{1}{2}(D + \Delta) + \tan\frac{1}{2}(D - \Delta)}{2};$$

$$r = \frac{\tan\frac{1}{2}(D + \Delta) - \tan\frac{1}{2}(D - \Delta)}{2}:$$

D being = 23° 27′ 49″, or nearly 23° 28′, the distance between the poles of the ecliptic and the equator; and Δ the polar distance of the parallel. When Δ > D, the sign of the second term will be changed.

45. The principal defect of this projection is the little resemblance and proportion between the arcs of the sphere and their projections. Thus, the arcs πA and AΠ represent arcs of 90°; πD represents an arc of 113° 28′, and Dπ, though greater, only represents 66° 32′; the arcs πu, πz, πH, &c. are of 90°; Dπ represents 23°

28′ = DAH = inclination of the ecliptic AHQ to the equator ADQ. It is true, however, that the greatest inequalities are *out* of the circle ADQB which is properly the projection. If we regard the circle πVΠXπ as a map of half the terrestrial globe, then πV, πY, πG, πE, will represent arcs of 90°, though πV will be the only one of those four which is actually a quadrant.

46. Another inconvenience of this projection, is the difficulty of finding the true distance of two points of which we have the projections. Yet, let M and N in the diagram to art. 42, be two such points: produce CM, CN, to *m* and *n* respectively; the arc *mn* will give us MCN which is the same as on the sphere. CM and CN are the tangents of the half distances from the pole of the projection. The spherical triangle will give (see chap. vi. equa. 2),

$$\cos \text{MN} = \cos \text{CM} \cos \text{CN} + \sin \text{CM} \sin \text{CN} \cos \text{C}$$

$$= \frac{(1-\tan^2 \tfrac{1}{2}\text{CM})(1-\tan^2 \tfrac{1}{2}\text{CN}) + 4 \tan \tfrac{1}{2}\text{CM} \tan \tfrac{1}{2}\text{CN} \cos \text{C}}{(1+\tan^2 \tfrac{1}{2}\text{CM})(1+\tan^2 \tfrac{1}{2}\text{CN})}$$

$$= \frac{(1-\text{CM}^2)(1-\text{CN}^2) + 4\text{CM} \,.\, \text{CN} \cos \text{C}}{(1+\text{CM}^2)(1+\text{CN}^2)}.$$

Take CM′ = CM and draw OM′M″, AM″ will be equal to the arc represented by CM. Proceed similarly for CN. Then cos MN may be computed from the above.

The third member of the equation is obtained from the second by substituting for cos CM, sin CM, their values in tan ½CM, &c. deduced by means of equa. R, chap. iv.

47. The projections here treated serving for the usual purposes of astronomy, we need not enter upon the explication of the other kinds of projection devised by geometers for different purposes. The principal of these is the *gnomonic projection*, in which the eye is supposed at the centre of the sphere, and the plane of projection a tangent plane to the sphere at any assumed point. All the points within adequate limits have their projections at the extremities of the *tangents* of their

distances from the point of contact; and those tangents form respectively the same angles as the arcs that measure the several distances from the principal point. In this projection too, a less circle will evidently be projected into an ellipse, a parabola, or a hyperbola, according as the distance of its most remote point is less, equal to, or greater than 90°, from the centre of the plane of projection.

But for a farther developement of these properties, and for the geometrical constructions derived from them, such as want to enter more minutely into this subject may consult *Emerson's Projection of the Sphere*, the treatise in *Bishop Horsley's Elementary Treatises on Practical Mathematics*, or that in the *Traité de Topographie, par Puissant.*

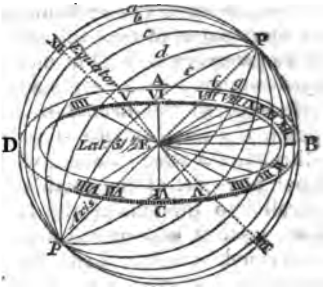
CHAPTER IX.

On the Principles of Dialling.

1. DIALLING, or *gnomonics*, is the art of drawing on the surface of any given body, whether plane, angular, or curved, a *sun-dial*, that is, a figure, the different lines of which, when the sun shines, indicate by the shadow of a style or gnomon the time of the day.

2. The general principles which serve as a basis to the theory of dialling, cannot be more aptly illustrated than they have been by Ozanam and Ferguson, in the following contrivance.

Suppose a hollow transparent sphere DPBp, of glass, to represent the earth as transparent, and its equator divided into 24 equal parts by so many meridian semicircles *a*, *b*, *c*, *d*, *e*, &c. one of which is the geographical meridian of any given place, as London, which it is supposed is at the point *a*; then if the hour of 12 were marked at the equator, both upon that meridian and the opposite one, and all the rest of the hours in order on the other meridians, those meridians would be the hour circles of London. Because, as the sun appears to move round the earth, which is in the centre of the visible heavens, in 24 hours, he will pass from one meridian to another in an hour. Then, if the sphere had an opaque axis, as PE*p*, terminating in the poles P and *p*, the shadow of the axis, which is in the same plane with the sun and with each meridian, would fall upon every particular meridian and hour when the sun came to the plane of the opposite meridian, and would consequently show the time at London, and at all other places on the same meridian. If this sphere were cut through the middle by a solid plane ABCD in the rational horizon of London, one half of the axis EP would be above the plane, and the other half below it; and if straight lines were drawn from the centre of the plane to those points where its circumference is cut by the hour circles of the sphere, such lines would be the hour lines of an horizontal dial for London: for the shadow of the axis would fall upon each particular hour line of the dial, when it fell upon the like hour circle of the sphere.



If the plane which intersects the sphere be imagined upright, and at the same time to face the meridian of the assumed place, the intersections of the several hour circles with the plane in this position, would give the hour lines of an erect, direct, south, or north dial.

And proceeding, in like manner, to contemplate planes, or other surfaces, any way posited in this hollow terrestrial sphere, we should by the several intersections of these surfaces with the hour circles, obtain the hour lines for every variety of dial. And since the earth itself compared with its distance from the sun, may, in reference to this branch of inquiry, be regarded as a point, if a small sphere of glass, or a small sphere constituted of a wire axis and wire hour circles, be placed upon any part of the earth's surface, so that its axis be parallel to the axis of the earth, and the sphere have lines upon it and planes within it, such as those above described, it will indicate the time of the day as accurately as if it were placed at the earth's centre, and the earth itself were as transparent as glass.

These general notions being premised, it will be proper to annex a few definitions.

3. The plane erected perpendicularly to the face of the dial, and the upper edge of which marks and bounds the shadow, is called the *gnomon;* the superior edge of the said plane is called the *style* of the dial, and it is always parallel to the earth's axis.

4. The line in which the plane of the gnomon intersects the plane of the dial is denominated the *substyle*.

5. The angle included between the style and the substyle is called the *elevation* of the style; in the following formulæ it will be denoted by E. In a horizontal dial this is, evidently, equal to the latitude of the place, or $E = L$.

6. While those dials whose planes are parallel to the plane of the horizon are called *horizontal dials;* such as have their planes perpendicular to the horizon are called *erect dials*.

7. Those erect dials whose planes are either parallel or perpendicular to the plane of the meridian, are called *direct erect dials;* they face one or other of the four cardinal points.

8. All other erect dials are called *declining dials*.

9. Those dials whose planes are neither parallel nor

perpendicular to the plane of the horizon, are called *inclining* or *reclining dials*. They may, at the same time, be either direct or declining, according as they present a sloping face to the cardinal points, or not.

10. The arch of the horizon which is intercepted between any given plane, and that of the prime vertical, is called the *declination* of the plane. It will be denoted by D, and will be regarded as *positive* when the declination is towards the north, *negative* when it is towards the south.

11. The *inclination*, I, of a plane is the angle which it makes with a vertical plane.

12. The intersection of the plane of the dial and that of the meridian passing through the style is called the *meridian* of the dial, or the hour line of 12.

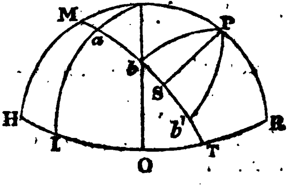
13. Those meridians passing through the style, which make angles of 15°, 30°, 45°, &c. with the meridian of the place (marking the hour line of 12) are called *hour circles*, and their intersections with the plane of the dial *hour lines*.

14. The angle formed by the substyle and the meridian is called the *horary angle of the substyle:* it will be denoted by M.

15. The angle included between the substyle and the horizon is termed the *inclination of the substyle:* it will be denoted by S.

GENERAL PROBLEM.

16. *To determine the requisites in a dial of any proposed inclination to a vertical plane, and declination from the prime vertical.*



Let HOR, in the marginal diagram, represent the horizon, HZR the meridian of the place, Z the zenith, ZO the prime vertical, P the elevated pole, P*b*, P*b′*, portions of hour circles, ZP*b*, ZP*b′*, corresponding hour angles, and let the plane on which it is proposed to draw the dial be coincident with the plane of the great circle $\mathrm{M}b\mathrm{T}$. Also, let $\mathrm{Z}a$ perpendicular to $\mathrm{M}a\mathrm{T}$ be drawn and produced to I. Then $\mathrm{PR} = \mathrm{L}$, the latitude of the place; $\mathrm{Z}a = \mathrm{I}$, the inclination of the plane; $\mathrm{OT} = \mathrm{HI} = \mathrm{D}$, its declination; PS drawn perpendicular to MT will be the position of the substyle; and $\mathrm{M}b$ is the inclination of the hour line to the meridian.

Now, we may regard the dial whose plane coincides with $\mathrm{M}a\mathrm{T}$, as a vertical or erect dial at the place whose zenith is M, where M and Z are on the same meridian, and, of course, reckon the hours alike. Let $\mathrm{MZ} = l$, and $\mathrm{D}' =$ comp. of $\mathrm{ZM}a$. Then $\mathrm{PZ} = 90° - \mathrm{L}$ would become $\mathrm{PM} = \mathrm{PZ} + \mathrm{ZM} = 90° - \mathrm{L} + l = 90° - (\mathrm{L} - l)$.

The right angled triangle $\mathrm{MZ}a$, gives

$$\tan \mathrm{M}a = \sin \mathrm{Z}a \tan \mathrm{MZ}a = \sin \mathrm{I} \tan \mathrm{D} \;\ldots\; (1.)$$

Here $\mathrm{M}a$ is the angle between the meridian and vertical, and is obviously evanescent when either I or D are.

When $\mathrm{I} = 90°$, $\mathrm{M}a$ becomes $= \mathrm{HI} = \mathrm{D}$.

$$\text{Also, } \tan \mathrm{ZM} = \tan l = \frac{\tan \mathrm{Z}a}{\cos \mathrm{MZ}a} = \frac{\tan \mathrm{I}}{\cos \mathrm{D}} \;\ldots\; (2.)$$

$$\cos \mathrm{ZM}a = \sin \mathrm{D}' = \sin \mathrm{D} \cos \mathrm{I} \;\ldots\; (3.)$$

$$\sin \mathrm{PS} = \sin \mathrm{E} = \sin \mathrm{ZM}a \sin \mathrm{PM} = \cos \mathrm{D}' \cos (\mathrm{L} - l) \;..\; (4.)$$

where E is the height of the style, or elevation of the pole above the plane.

Then, $\mathrm{P}b$ being any hour circle whatever, the spherical triangle $\mathrm{P}b\mathrm{M}$, gives us, from equa. (4) of spherical trigonometry [chap. vi. 23].

$$\cot \mathrm{M}b \sin \mathrm{PM} = \cos \mathrm{PM} \cos \mathrm{PM}b + \sin \mathrm{PM}b \cot \mathrm{P}.$$

But, $\cos \mathrm{PM}b = \sin \mathrm{D}'$, and $\sin \mathrm{PM}b = \cos \mathrm{D}'$;

also, $\frac{\cos}{\sin} = \cot$. Therefore, dividing by $\sin \mathrm{PM}$,

$$\cot \mathrm{M}b = \sin \mathrm{D}' \tan (\mathrm{L} - l) + \frac{\cos \mathrm{D}'}{\cos (\mathrm{L} - l)} \mathrm{P} \;\ldots\; (5.)$$

$$\tan \mathrm{MT} = \tan (\mathrm{M}a + 90°) = - \cot \mathrm{M}a,$$

where MT is the inclination of the horizontal plane to the meridian.

The above expression (5) is general, and by no means complex. It may, however, be rendered more convenient for further deductions and corollaries by exterminating l and D'. Thus, for the first term,

$$\sin \mathrm{D}' \tan (\mathrm{L} - l) = \sin \mathrm{D} \cos \mathrm{I} \, \frac{\tan \mathrm{L} - \tan l}{1 + \tan \mathrm{L} \tan l}$$

$$= \left[\sin \mathrm{D} \cos \mathrm{I} \left(\tan \mathrm{L} - \frac{\tan \mathrm{I}}{\cos \mathrm{D}}\right) \div \left[1 + \frac{\tan \mathrm{I}}{\cos \mathrm{D}} \tan \mathrm{L}\right]\right.$$

$$= \frac{\sin \mathrm{D} \cos \mathrm{I} \tan \mathrm{L} - \sin \mathrm{I} \tan \mathrm{D}}{1 + \tan \mathrm{I} \sec \mathrm{D} \tan \mathrm{L}}.$$

And, for the second term,

$$\frac{\cos \mathrm{D}'}{\cos (\mathrm{L} - l)} = \frac{\sin \mathrm{PM}b}{\cos (\mathrm{L} - l)} = \frac{\sin \mathrm{I}}{\sin l (\cos \mathrm{L} - l)} =$$

$$\frac{\sin \mathrm{I}}{\sin l \cos l \cos \mathrm{L} + \sin^2 l \sin \mathrm{L}} = \frac{\sin \mathrm{I}}{\cos^2 l (\tan l \cos \mathrm{L} + \tan^2 l \sin \mathrm{L})}$$

$$= \frac{\sin \mathrm{I} (1 + \tan^2 l)}{\tan l \cos \mathrm{L} + \tan^2 l \sin \mathrm{L}}$$

$$= \left[\sin \mathrm{I} \left(1 + \frac{\tan^2 \mathrm{I}}{\cos^2 \mathrm{D}}\right)\right] \div \left[\frac{\tan \mathrm{I} \cos \mathrm{L}}{\cos \mathrm{D}} + \frac{\tan^2 \mathrm{I}}{\cos^2 \mathrm{D}} \sin \mathrm{L}\right]$$

$$= \frac{\sin \mathrm{I} (\cos^2 \mathrm{D} + \tan^2 \mathrm{I})}{\tan \mathrm{I} \cos \mathrm{D} \cos \mathrm{L} + \tan^2 \mathrm{I} \sin \mathrm{L}} = \frac{\sin \mathrm{I} (1 - \sin^2 \mathrm{D} + \tan^2 \mathrm{I})}{\tan \mathrm{I} \cos \mathrm{D} \cos \mathrm{L} + \tan^2 \mathrm{I} \sin \mathrm{L}}$$

$$= \frac{\sin \mathrm{I} (\sec^2 \mathrm{I} - \sin^2 \mathrm{D})}{\tan \mathrm{I} \cos \mathrm{D} \cos \mathrm{L} + \tan^2 \mathrm{I} \sin \mathrm{L}} = \frac{\cos \mathrm{I} (\sec^2 \mathrm{I} - \sin^2 \mathrm{D})}{\cos \mathrm{D} \cos \mathrm{L} + \tan \mathrm{I} \sin \mathrm{L}}$$

$$= \frac{\cos^2 \mathrm{I} (\sec^2 \mathrm{I} - \sin^2 \mathrm{D})}{\cos \mathrm{I} \cos \mathrm{D} \cos \mathrm{L} + \sin \mathrm{I} \sin \mathrm{L}} = \frac{1 - \cos^2 \mathrm{I} \sin^2 \mathrm{D}}{\cos \mathrm{L} \cos \mathrm{I} \cos \mathrm{D} + \sin \mathrm{I} \sin \mathrm{L}}.$$

The last values of $\sin \mathrm{D}' \tan (\mathrm{L} - l)$ and of $\frac{\cos \mathrm{D}'}{\cos (\mathrm{L} - l)}$, being substituted for them in equa. (5), it becomes

$$\cot \mathrm{M}b = \frac{\sin \mathrm{D} \cos \mathrm{I} \tan \mathrm{L} - \sin \mathrm{I} \tan \mathrm{D}}{1 + \tan \mathrm{I} \sec \mathrm{D} \tan \mathrm{L}}$$

$$+ \frac{1 - \cos^2 \mathrm{I} \sin^2 \mathrm{D}}{\cos \mathrm{L} \cos \mathrm{I} \cos \mathrm{D} + \sin \mathrm{I} \sin \mathrm{L}} \cot \mathrm{P} \ldots\ldots (6.)$$

This is the general expression for the inclination of the hour line to the meridian: the first term is constant; the second has a constant coefficient to the variable quantity cot P, which undergoing all its changes of *magnitude* in 90° = 6 × 15°, or 6 hours, the tracing of 6 hours of the dial will serve for tracing the rest.

For the horary angle of the substyle we have

$$\tan \mathrm{MS} = \cos \mathrm{PMS} \tan \mathrm{PM} = \sin \mathrm{D}' \cot (\mathrm{L} - l),$$

which, on comparison with the value of $\sin \mathrm{D}' \tan (\mathrm{L} - l)$ above obtained, evidently reduces to

$$\tan \mathrm{MS} = \frac{\sin \mathrm{D} \cos \mathrm{I} \cot \mathrm{L} + \sin \mathrm{I} \tan \mathrm{D}}{1 - \tan \mathrm{I} \sec \mathrm{D} \cot \mathrm{L}} \ldots\ldots (7.)$$

Farther, since $\cos \mathrm{D}' \cos (\mathrm{L} - l) = \sin \mathrm{PMS} \sin \mathrm{PM}$ $= \frac{\sin \mathrm{I}}{\sin l} (\cos \mathrm{L} \cos l + \sin \mathrm{L} \sin l) = \sin \mathrm{I} (\cos \mathrm{L} \cot l$ $+ \sin \mathrm{L})$

the equation (4) becomes

$$\sin \mathrm{E} = \sin \mathrm{I} (\sin \mathrm{L} + \cos \mathrm{L} \cos \mathrm{D} \cot \mathrm{I}) = \sin \mathrm{I} \sin \mathrm{L} + \cos \mathrm{I} \cos \mathrm{D} \cos \mathrm{L} \ldots\ldots (8.)$$

From these theorems (6), (7), (8), the whole of dialling on planes may be deduced.

17. Thus, let I = 0, or the plane become erect; then from equa. (6), we have

$$\cot \mathrm{M}b = \tan \mathrm{L} \sin \mathrm{D} + \frac{\cos \mathrm{D}}{\cos \mathrm{L}} \cot \mathrm{P} \ldots\ldots (9.)$$

In this hypothesis equa. (7) becomes

$$\tan \mathrm{MS} = \cot \mathrm{L} \sin \mathrm{D} = \cot s \ldots\ldots (10.)$$

By which the angle between the substyle and the horizon is determined.

And equa. (8) becomes

$$\sin \mathrm{E} = \cos \mathrm{L} \cos \mathrm{D} \ldots\ldots (11.)$$

We have also, when M and Z coincide,

$$\cot \mathrm{ZPS} = \sin \mathrm{L} \cot \mathrm{D} \ldots\ldots (12.)$$

Thus we obtain the horary angle of the substyle.

18. If D as well as I become = 0, then

$$\cot \mathrm{M}b = \frac{\cot \mathrm{P}}{\cos \mathrm{L}} = \cot \mathrm{P} \sec \mathrm{L};$$

$$\text{consequently } \frac{1}{\cot \mathrm{M}b} = \frac{1}{\cot \mathrm{P} \sec \mathrm{L}};$$

$$\text{or, } \tan \mathrm{M}b = \tan \mathrm{P} \cos \mathrm{L} \ldots\ldots (13.)$$

In this hypothesis also, we have

sin E = cos L . . . . (14.) . . . . . cot S = 0 . . . . (15.)

The three last formulæ evidently serve for erect direct *south* or *north* dials; in which equa. (15) shows that the substyle is perpendicular to the horizon.

19. If while I = 0, D become = 90°;

then will equa. (10) be transformed to cot S = cot L (16),

and equa. (11) to sin E = 0 . . . . (17.)

This is the case of the erect direct *east* or *west* dial, in which it appears that the style is parallel to the plane, and the substyle inclined to the horizon in an angle equal to the latitude.

Here, since the style is parallel to the plane, the dial has no centre; all the hour lines, therefore, are parallel to the substylar line, which is the hour line of 6 o'clock: the respective distances of the hour lines from the substylar are measured by *a* cot P, on a perpendicular to the 6 o'clock hour line; or by $\frac{a \cot P}{\cos L}$ on a vertical line. This gives an infinite distance between the 6 o'clock and 12 o'clock hour lines; as there manifestly ought to be, because at noon the solar rays will be perpendicular to the plane of the dial.

20. If when D = 0, we have I = 90°, the equations (6) and (8) become

$$\cot Mb = \frac{\cot P}{\sin L}, \text{ whence } \frac{1}{\cot Mb} = \frac{\sin L}{\cot P};$$

or, tan M*b* = sin L tan P . . . . (18.)

and sin E = sin L . . . . (19.)

These theorems obviously apply to the horizontal dial.

21. If, in the last hypothesis, L become equal to 90°, we shall have

tan M*b* = tan P . . . . sin E = radius.

This would apply to a horizontal dial at the poles, or a dial in any latitude with its face posited *parallel to the equator*. Here the formulæ show that the style would become a *pin* placed perpendicularly to the centre of the dial, round which the hour lines would be radii drawn from the foot of the style to make angles of 15°, 30°, 45°, &c. with the meridian, or hour-line of 12.

PROBLEM II.

22. *To describe a horizontal dial for any proposed latitude.*

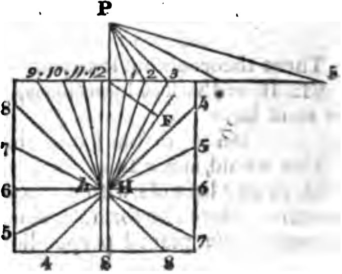
This is the simplest dial to draw next to the polar dial, just mentioned; and it is the most useful, because, if it be posited where the sun's rays meet with no obstructions, that luminary will shine upon it from his rising to his setting. The theorems from which the construction is to be deduced are,

$$\tan Mb = \sin L \tan P, \text{ and } \sin E = \sin L.$$

Here Mb is the measure of the angle H, between the 12 o'clock and any other hour line on the dial, and P the hour angle from the meridian, as it varies at the poles of the heavens; the latter, therefore, varies uniformly, while the former only varies uniformly in certain limited cases; as, for example, in the horizontal dial at the latitude of 90°. In any other latitude, making the terms of the equation homogeneous, we have

$$\sin 90^\circ \tan H = \sin L \tan P.$$

Hence, if two radii be assumed in the ratio of sin 90° to sin L, tan H referred to the former radius will always be equal to tan L referred to the latter. From this consideration the following construction is deduced:



On the proposed plane assume the right line 12 HS for the meridian, or 12 o'clock hour line, parallel to which draw the line 12 *h*s at a distance equal to the proposed thickness of the style. Perpendicularly to these draw 6H6, for the east and west line of the dial, or the 6 o'clock hour line. Make the angle 12HF = the latitude of the place, and from 12 let fall the perpendicular 12F upon HF. Make 12P, upon H12 prolonged, equal to 12F. From P draw lines P1, P2, P3, &c. (to terminate in the line 12-5, perpendicular to 12H) and to make angles 12P1, 12P2, 12P3, &c. equal to 15°, 30°, 45°, &c. Then from the centre H draw H1, H2, H3, H4, H5, for the hour lines of I, 2, 3, 4, and 5, in the afternoon. Take, on the other side of the substylar line 12-11, 12-10, 12-9, &c. respectively equal to 12-1, 12-2, 12-3, &c. and from *h* draw the lines *h*7, *h*8, *h*9, &c. Produce the lines 4H, 5H, for the lines of 4 and 5 in the morning; and produce the lines 7*h*, 8*h*, for the hour lines of 7 and 8 o'clock in the afternoon.

The truth of this construction is manifest from the remarks which precede it. For 12P = 12F is evidently the sine of the latitude to the radius H12. And while 12-1, 12-2, 12-3, &c. are the tangents of 15°, 30°, 45°, &c. respectively, to the radius P12; the same lines are tangents to the angles 12H1, 12H2, 12H3, &c. Consequently, while the former are hour angles at the pole, the latter are the corresponding hour angles at the centre of the dial.

The quarter and half hour lines are drawn by setting off angles of 3° 45′, 7° 30′, 11° 25′, &c. from the meridian line: but they are omitted in the diagram to prevent confusion.

The angle 12HF is also equal to the elevation of the style; for E = L.

As to the gnomon, it should be a metallic triangle of the thickness *h*H, and having one angle = L the latitude. It must be fixed perpendicularly to the plane, on the space left for it in the figure from *h*H towards 12, and having its angle L at *h*H: then will the style of the dial be parallel to the earth's axis.

PROBLEM III.

23. *To describe an erect direct south dial for any proposed north latitude, or an erect direct north dial, for any proposed south latitude.*

Here the formulæ are those numbered 13, 14, and 15, of which the two first are

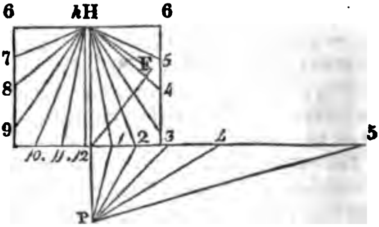
$$\tan Mb = \cos L \tan P, \text{ and } \sin E = \cos L.$$

Substituting H for Mb in the former of these, and making the terms homogeneous, we have

$$\sin 90^\circ \tan H = \cos L \tan P.$$

This equation is similar to the equation of the hour angles in the horizontal dial; and the construction will therefore be similar, except that cos L is here to be employed instead of sin L.

On the proposed plane draw the right line 6H6, for the east and west line of the dial, or the hour line of 6. In the middle of this line set off hH equal to the proposed thickness of the gnomon, and through h, H, draw lines perpendicular to 6H6 to terminate in the line 10-5, parallel to 6H6 at a convenient distance. Draw HF to make the angle 12HF equal to the latitude of the place for which the dial is made, and let fall upon HF the perpendicular 12F. Make 12P = HF, the cosine of the angle 12HF to radius H12: draw from P lines P1, P2, &c. to make angles of 15°, 30°, &c. with P12: draw lines from H to meet these in the line 10-5; set off corresponding lines on the left side of the dial, and the construction is completed. The demonstration is the same as in prob. 2.



The angle made between the style and substyle is here equal to the *complement* of the latitude. When the dial is placed vertically to face the proper cardinal point, the line 6H6 will be horizontal, and the style sloping downwards from H at an angle of $90^\circ - L$ will again be parallel to the earth's axis, as it ought to be.

Note. An erect north dial for a place in north latitude is constructed in exactly the same manner as an erect south dial; but the *position* of the dial must be reversed. In the case of the north dial for north latitude the line 6A6, instead of being the top will become the bottom of the dial.

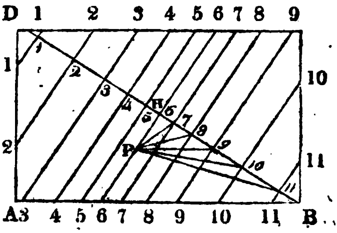
The same may be observed in reference to an erect south dial for a south latitude.

PROBLEM IV.

24. *To describe an erect direct east dial, for any proposed north latitude, or an erect direct west dial for any south latitude.*

It appears from art. 19, that the substyle in this dial will make an angle with the horizon equal to the latitude, that the hour lines will be all parallel to the substyle, and at the distances indicated by a cot P, a being the height of the gnomon, and P the hour angle from the meridian at the pole of the sphere.

Let AB be assumed as the horizontal line on the proposed dial. From the corner B draw BD to make an angle ABD = the complement of the latitude, and about the middle, H, of that line draw perpendicularly to it the line 6H6 for the 6 o'clock hour line: this will, also, be the substylar line, and will evidently make with the horizon an angle B6H = the latitude of the place. Assume any point, as that marked 11, for the point where the 11 o'clock



hour line is to intersect the line BH; and draw 11P to make the angle H11P = 15°: so shall PH be the height of the gnomon. Set off angles HP7, HP8, HP9, &c. respectively equal to 15°, 30°, 45°, &c. and through the points where the lines P7, P8, &c. intersect the line BD, draw lines parallel to PH: also, set off, from H towards D, H5 = H7, H4 = H8, &c. and through the points 5, 4, 3, &c. draw other lines parallel to PH, these shall be the hour lines required.

The truth of the construction is evident from what has preceded.

A rectangular gnomon of altitude = PH, being set up perpendicularly to the plane of the dial, will, when the dial itself is posited vertically to face the east, have its upper edge parallel to the earth's axis. It is here supposed that the upper edge of the gnomon is reduced to a mere line: if it have any measureable thickness, allowance must be made for it in the construction, as in the preceding problems.

Note. An erect direct west dial for any place in north latitude, may be constructed exactly in the same manner as is just taught, except that, instead of beginning the construction from the right hand, B, of the plane, the operation must commence at the left hand, A, and that the figures expressing the hours of 1, 2, 3, &c. in the afternoon, be placed between A and the hour line of 6.

The like may be observed with regard to an erect direct east dial for any south latitude.

PROBLEM V.

25. *To describe an erect south dial for any proposed north latitude, to have a given declination from the west; or an erect north dial for a south latitude, to decline from the east.*

The formulæ in the general problem which are applicable to the present, are those numbered (9), (10), and (11), viz.

$$\cot Mb, \text{ or, } \cot H = \tan L \sin D + \frac{\cos D}{\cos L} \cot P,$$

for the determination of the hour lines;

$$\cot S = \cot L \sin D,$$

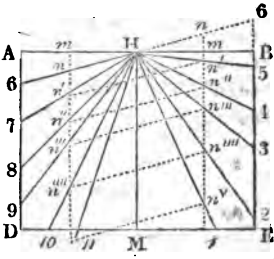
for that of the angle between substyle and horizon;

$$\text{and } \sin E = \cos L \cos D,$$

for the elevation of the style.

From these and a table of natural sines and tangents, the hour lines, &c. may easily be drawn, thus:

Let ABCD be the plane on which the dial is to be constructed. Assume H for the centre of the hour lines, and the vertical line HM for the meridian, or 12 o'clock hour line. Now, since the only variable quantity in the preceding expression for cot H, the cotangent of the angle between the meridian of the plane and any hour line, is cot P, the cotangent of the corresponding hour angle at the pole, we may, by taking that = 0, get a fixed hour line to which the others may be referred. Thus, the expression, when the hour angle at the pole is $90° = 6 \times 15°$, becomes



$$\cot MH6 = \tan AH6 = \tan L \sin D;$$

an equation by which the position of the 6 o'clock hour line may be readily determined. For, if Hm be taken between H and A, equal to the unit of any measure, as one inch, one foot, &c. we shall have

$$mn = \tan L \sin D, \text{ in the same measure;}$$

the product being at once determinable by means of a table of sines and tangents. Through H and n draw a line both ways, to meet AD and the prolongation of EB; then that portion of this right line which falls on the left hand side of the plane will be the hour line of 6 in the morning, while the other portion to the right of H, and above the plane, would be the hour line of 6 in the afternoon: that, however, is useless in this dial.

To ascertain the position of the other hour lines, we have only to estimate the remaining term of the equation by the several corresponding values of $\cot P$ for the different hours from 12 or 6, and set those values from n to n', n to n'', n to n''', &c. on the line $mnn'n''$ perpendicular to AB. These values are successively

$$\frac{\cos D}{\cos L} \cot 75^\circ = \cos D \sec L \tan 15^\circ$$

$$\frac{\cos D}{\cos L} \cot 60^\circ = \cos D \sec L \tan 30^\circ$$

$$\frac{\cos D}{\cos L} \cot 45^\circ = \cos D \sec L \tan 45^\circ = \cos D \sec L$$

$$\frac{\cos D}{\cos L} \cot 30^\circ = \cos D \sec L \tan 60^\circ$$

$$\frac{\cos D}{\cos L} \cot 15^\circ = \cos D \sec L \tan 75^\circ \text{ (see chap. iv. 41.)}$$

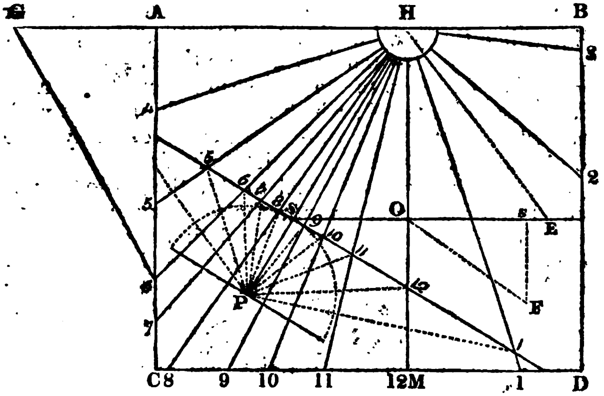
These being computed and set off, the several right lines $Hn'7$, $Hn''8$, $Hn'''9$, &c. drawn from H through the points n', n'', n''', &c. will be the morning hour lines required. The afternoon hour lines may be computed and set off by the same formulæ. Indeed, taking the other vertical line $mn'n''n'''$, &c. between H and B, ($Hm = Hm$) the several values of mn, nn', nn'', &c. will be numerically the same, because the corresponding values of $\cot P$ are so; the only change being in the *sign* from $+$ to $-$, on account of passing the meridian.

Hence, since the values of nn', nn'', nn''', &c. on the two parallel vertical lines are respectively equal, lines joining their extremities will form a series of parallelograms: and hence, the morning hour lines being drawn, the evening hour lines equally distant from 6 o'clock, may be determined by drawing $n'n'$, $n''n''$, $n'''n'''$, &c. parallel to $6H6$, and then drawing through n', n'', n''', to the right, the lines $Hn'5$, $Hn''4$, $Hn'''3$, &c.

Thus, it appears, as has been observed by *Ozanam*, *Emerson*, and *Delambre*, that in all plane dials, one half of the dial being traced, determines the other by the several distances from the hour line of 6, measured on two vertical lines equidistant from the meridian. Thus these distances are equal between the lines from
6 to 7 in the morning and from 5 to 6 in the evening,
6 to 8 …… morning ……… 4 to 6 ….. evening,
&c. &c. &c.

The position of the substyle, in this instance between HM and HA, is determined by the preceding equation for cot s, and the elevation of the style by the preceding theorem for sin E.

26. But this kind of dial, as well as horizontal and direct dials, may be constructed independent of computation. Thus:



On the proposed plane ABCD assume H for the centre from which the hour lines shall diverge, and draw the vertical line HM for the hour line of 12. Produce HA to G, so that HA shall be to AG, in the ratio of sin 90° to sin D; draw G6 to make the angle AG6 = L, the latitude of the place, and join H6, which will be the 6 o'clock hour line. From H draw HE parallel to O6 to meet a line OE parallel to AB in E. From O the intersection of OE and HM, draw OF to make the angle EOF = complement of the given declination; make OF = OE, draw F*s* perpendicular to OE, make OS = O*s*, and join HS which will be the substylar line. Perpendicular to HS draw the right line 5S1, intersecting the meridian in 12, and the 6 o'clock hour line in 6. On 6-12 as a diameter describe a semicircle to cut HS produced in P. With centre P and radius PS describe a semicircle, which divide into arcs of 15° each, both ways from the line P6, or, which amounts to the same, set off angles 6P7, 7P8, 8P9, 9P10, &c. each = 15°. Through the points 5, 6, 7, 8, &c. where the lines bounding these angles cut the lines 5S1, draw from H the lines H44, H55, H66, H77, &c. they will be the hour lines required.

For, first, since the expression for the 6 o'clock hour line, when the terms are rendered homogeneous, is

$$\sin 90^\circ \tan AH6 = \sin D \tan L,$$

we shall have tan L to the radius sin D equal to tan AH6 to the radius sin 90°; which is obviously the case with regard to the triangles GA6, HA6. Consequently, the 6 o'clock hour line is rightly determined.

Again, since EH is parallel to 6G, the angle OEH is equal to the latitude, and EHO the co-latitude; therefore OE = OF, is the cotangent of the latitude to the radius OH; and since EOF is equal to the co-declination, O*s* = OS is the sine of the declination to the radius OF = OE; or O*s* = OS = sin D cot L. But OS = tan OHS = cot S (substylar angle with horizon) to radius HO: therefore cot S = sin D cot L, as it ought to be; and the substyle is rightly determined.

But the substylar line is evidently a portion of the right line between the centre of the dial and the apparent pole: therefore, the apparent pole, P, lies in the prolongation of HS. At the apparent pole, P, also the horary angle between 6 o'clock and 12, is 6 × 15°, or a right angle; and the angle in a semicircle is a right angle: therefore, the semicircle described upon 6s12 as a diameter, will intersect the prolongation of HS in P, the apparent pole.

The truth of the remainder of the construction is manifest*.

27. For astronomical methods of determining the meridian, and the declination of any vertical plane, the student may turn to prob. 2 of the next chapter, examples 3, 4, and 5.

CHAPTER X.

Astronomical Problems.

1. SINCE the science of astronomy has given birth to spherical trigonometry, it is to be expected that at least some elementary problems in astronomy may be admitted into an introduction like the present. To determine the position of points in the apparent heavens, astronomers first referred to two planes, the *horizon* and the *meridian* (see chap. viii. § 1), which are fixed in reference to any one place on the surface of the earth. But the necessity of comparing observations at different

---

* The problems given in this chapter will suffice to show the application of the principles of dialling to the most useful cases. They who wish to pursue either the theory or the mechanical part of dialling, through all its modifications, may consult *Leadbetter's Mechanic Dialling*, *Emerson's Dialling*, the treatise on dialling in the 5th vol. of *Ozanam's Course of Mathematics*, and that in the 3d vol. of *Dr. Hutton's* edition of *Ozanam's Recreations*. There is also an elegant essay on dialling by M. de Parcieux, at the end of his *Trigonometrie Rectiligne et Spherique;* and a neat and simple deduction of the practice of dialling from the principles of perspective, at the end of *S'Gravesande's Essay on Perspective.*

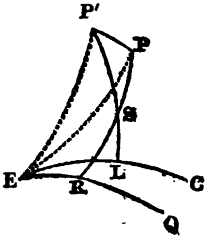
places, has led to the introduction of other planes and circles into the science, independent of the position of the observers, and even of the figure of the earth. Thus, when the situation of the celestial *equator*, and the manner of valuing the angles between meridians by the measure of time, became known, they were employed to determine the position of the heavenly bodies by means of their *right ascension* and their observed *declination*. Afterwards, as it was found that a considerable portion of the celestial phenomena relative to the planetary system, occur in the plane of the *ecliptic*, or in planes but little inclined to it, it was found expedient to refer the position of the stars to the same plane, that is, to determine their *latitude* and *longitude* (chap. viii. art. 17, 18).

These, and many other branches of astronomical inquiry, which we shall not here be able to touch, depending upon the mutual relations and intersections of different *circles of the sphere*, fall necessarily within the department of trigonometry. A few only will here be selected.

PROBLEM I.

2. Given the obliquity of the ecliptic, and either the right ascension and declination of a star, or its latitude and longitude, to find the other two, and the angle of position.

Let EC in the annexed figure be a portion of the ecliptic, EQ a portion of the equator, the two circles intersecting in ♈ the first point of Aries, in an angle, i, of 23° 27′ 49″. Let P′ be the elevated pole of the ecliptic, P that of the equator, PP′ a portion of the solstitial colure. From P′ and P let quadrants P′SL, PSR, of great circles, be drawn through S, the place of the star. Then ER



will be the right ascension, a, of the star, SR its declination, d, or PS, its co-declination, EL its longitude, l, SL its latitude, L, or PS its co-latitude, $PSP' = p$, the angle of position; and PP′ is given = 23° 27′ 49″, the present measure of the obliquity of the ecliptic. It is farther evident that SP′P is the complement of the longitude, and P′PS = P′PE + EPR = 90° + right ascension; as indicated at the poles of their respective circles.

Now, if they are the right ascension and declination which are supposed known, in addition to the obliquity, we shall have, from the triangle SPP′, (see chap. vi. equa. 2 and 4),

$$\cos P'S = \sin PP' \sin SP \cos P'PS + \cos P'P \cos SP,$$

$$\cot PP'S = \frac{\cot SP \sin PP' - \cos P'PS \cos P'P}{\sin P'PS}.$$

Adopting the preceding literal representatives of these sides and angles, and remembering that $\cos P'PS = \cos(90° + a) = -\sin a$, these become

$$\sin L = -\sin i \cos d \sin a + \cos i \sin d \ldots\ldots (1.)$$

$$\left.\begin{aligned} \tan l &= \frac{\tan d \sin i + \sin a \cos i}{\cos a} \\ &= \tan d \sin i \sec a + \tan a \cos i \end{aligned}\right\} (2.)$$

These two formulæ may be accommodated to logarithmic computation, by taking a subsidiary angle ϕ such that

$$\tan \phi = \frac{\sin a}{\tan d}:$$

for then, exterminating $\sin a$ from the former, and $\tan d$ from the latter, we shall have

$$\sin L = \frac{\sin d \cos(\phi + i)}{\cos \phi} \ldots\ldots (3.)$$

$$\tan l = \frac{\tan a \sin(\phi + i)}{\sin \phi} \ldots\ldots (4.)$$

3. If, on the contrary, the latitude and longitude of the star are given, we shall have the declination and

right ascension by the following theorems similarly deduced, viz.

$$\cos \text{PS} = \sin \text{P}'\text{P} \sin \text{P}'\text{S} \cos \text{PP}'\text{S} + \cos \text{PP}' \cos \text{P}'\text{S},$$

$$\cot \text{P}'\text{PS} = \frac{\cot \text{P}'\text{S} \sin \text{P}'\text{P} - \cos \text{PP}'\text{S} \cos \text{P}'\text{P}}{\sin \text{PP}'\text{S}}:$$

where again, introducing the literal values and observing that $\cot \text{P}'\text{PS} = -\tan a$, we have

$$\sin d = \sin i \cos \text{L} \sin l + \cos i \sin \text{L} \dots . (5.)$$

$$\left.\begin{aligned} \tan a &= \frac{-\tan \text{L} \sin i + \sin l \cos i}{\cos l} \\ &= -\tan \text{L} \sin i \sec l + \tan l \cos i \end{aligned}\right\} (6.)$$

These equations are obviously analogous to those we have just deduced for the latitude and longitude, the only difference being that here the obliquity enters the formulæ *negatively*. Hence by taking another subsidiary angle φ', so that

$$\tan \varphi' = \frac{\sin l}{\tan \text{L}},$$

and eliminating as before, there will result,

$$\sin d = \frac{\sin \text{L} \cos (\varphi' - i)}{\cos \varphi'} \dots . (7.)$$

$$\tan a = \frac{\tan l \sin (\varphi' - i)}{\sin \varphi'} \dots . (8.)$$

4. As for the angle of position $\text{PSP}' = p$, it is easily found from the relation between the sines of angles and the sines of their opposite sides; for from hence we have

$$\sin p = \frac{\sin i \cos a}{\cos \text{L}}, \text{ and } \sin p = \frac{\sin i \cos l}{\cos d} \dots . (9.)$$

5. We have also, from the same consideration,

$$\cos a \cos d = \cos \text{L} \cos l \dots . (10.)$$

And when $\text{L} = 0$, as is always the case with the sun, we have

$$\cos a = \frac{\cos l}{\cos d} = \cos l \sec d \dots . (11.)$$

$$\tan a = \tan l \cos i \dots . (12.)$$

$$\cos l = \cos a \cos d \dots . (13.)$$

$$\sin d = \sin l \sin i \ldots\ldots (14.)$$
$$\sin l = \sin d \operatorname{cosec} i \ldots (15.)$$

The preceding formulæ have been deduced upon the supposition that the heavenly body has not gone beyond the first quadrant of right ascension from the vernal equinox. But they are applicable to all positions by simply regarding the mutations of signs in the several sines, cosines, tangents, &c. according to the arcs to which they refer. The right ascensions and longitudes, being reckoned from the first point of Aries through their respective circles, will at once indicate by their attendant signs, + or −, to which quadrant they belong. And as to the declinations and latitudes, they being regarded as positive when toward the elevated pole, will be regarded as negative when towards the contrary pole. Thus, with us, north latitudes and declinations will be positive, south latitudes and declinations, negative.

Example I.

The right ascension of Aldebaran being 67° 40′ 30″, its declination 16° 8′ 20″ N. Required the longitude, latitude, and angle of position.

Here, taking φ so that $\tan \varphi = \frac{\sin a}{\tan d}$, and substituting sec φ, cosec φ, and sec d, for $\frac{1}{\cos \varphi}$, $\frac{1}{\sin \varphi}$, and $\frac{1}{\cos d}$, in equa. 3, 4, and 9, they become

$$\sin \text{L} = \sin d \cos (\phi + i) \sec \varphi,$$
$$\tan l = \tan a \sin (\phi + i) \operatorname{cosec} \varphi,$$
$$\text{and } \sin p = \sin i \cos l \sec d:$$

the log. operations corresponding to which will be as follows:

From log sin a . . 67° 40′ 30″ . . . . 9·9661625
Take . . . tan d . . 16° 8′ 20″ . . . . 9·4614544

Remains tan φ . . 72° 37′ 46″ . . . . 10·5047081

Then, to find L the latitude, add together,

Log sin d	16° 8′ 20″	9·4439927
cos $(\phi + i)$	96° 5′ 35″	9·0258936
sec φ	72° 37′ 46″	10·5249824
The sum sin L	5° 40′ 18″ S.	8·9948687

Here, because cos $(\phi + i)$ being in the second quadrant is negative (chap. iv. art. 4), and the other terms are positive, the product is negative, and therefore the latitude is *south*.

Next, to find l the longitude, add together,

Log tan a	67° 40′ 30″	10·3865391
sin $(\phi + i)$	96° 5′ 35″	9·9975396
cosec φ	72° 37′ 46″	10·0203305
The sum tan l	68° 29′ 28″	10·4044092

Here all the terms being positive, their product is positive; conseq. the longitude is in the first quad.

Lastly, for p the angle of position, add together,

Log sin i	23° 27′ 49″	9·6000647
cos l	68° 29′ 28″	9·5642090
sec d	16° 8′ 20″	10·0174616
The sum sin p	8° 44′ 26″	9·1817353

Example II.

What are the latitude and longitude of the moon, when her right ascension is 304° 21′, and declination 22° 57′. S?

Ans. Longitude 10^s 1° 21′ 54″, latitude 3° 8′ 46″ S.

Example III.

When the longitude of the moon is 1^s 7° 41′ 23″, and the latitude 3° 49′ 57″ S., what are the right ascension and declination?

Ans. Right ascension 36° 36′, declination 10° 28′ N.

Example IV.

When the sun's declination is 19° 18′ 20″ N. what are his right ascension and longitude?

First, to find the longitude, employ equa. 15.

that is, $\sin l = \sin d \operatorname{cosec} i$.

To log sin d.... 19° 8′ 20″.... 9·5156873
Add log cosec i.. 23° 27′ 49″.... 10·3999353

The sum is log sin l... 55° 25′ 43″.... 9·9156226

Here since the declination is north, or positive, sin l is positive, and lies, therefore, either in the first or second quadrant, l is, therefore, either 55° 25′ 43″ (that is, 1^s 25° 25′ 43″) as above expressed, or its supplement 124° 34′ 17″, that is, 4^s 4° 34′ 17″.

To find the right ascension, take equa. 11,

that is, $\cos a = \cos l \sec d$.

To log cos l.. 55° 25′ 43″..... 9·7539143
Add log sec d.. 19° 8′ 20″.... 10·0246938

The sum is log cos a.. 53° 5′ 6″.... 9·7786081

This reduced to time at the rate of 15° to an hour, gives $3^h\ 32^m\ 20^s$, for the right ascension corresponding to the longitude 55° 25′ 43″. If the other longitude had been taken, the resulting right ascension would have been $8^h\ 27^m\ 40^s$, the supplement of the former to 12 hours.

The days that correspond to these in the Nautical Almanac for 1816, are May 16 and July 27.

Example V.

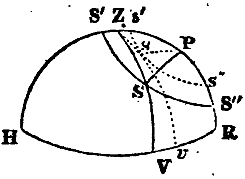
Given the sun's longitude 6^s 8° 9′ 36″, to find the right ascension, declination, and time in 1816.

Ans, $12^h\ 29^m\ 58^s$, right ascension, 3° 14′ 24″ S. declin. time Oct. 1, 1816, at noon.

PROBLEM II.

6. To investigate formulæ that shall be applicable to inquiries in reference to the times of risings and settings of the heavenly bodies, the azimuths, the duration of twilight, &c.

Let HZR be the meridian of the place of observation, HVR the horizon, Z the zenith, P the elevated pole, S the place of the sun, a fixed star, or other heavenly body, S′S S″ part of the circle it appears to describe about the pole P during the earth's diurnal rotation, ZSV part of the vertical circle on which the body is at any proposed time, and PS a portion of a great circle passing through the pole and the place of the body. Then PS is $= 90^\circ - d$ the co-declination; $\text{PZ} = 90^\circ - l$, the co-latitude; $\text{ZS} = n$, the zenith distance, or $= 90^\circ - a$ the co-altitude; $\text{ZPS} = \text{P}$ the horary or polar angle between the two meridians ZP, SP; and $\text{SZP} = \text{Z}$, the azimuth, measured also by the arc VR. Then the formula we may first employ in the present investigation, is equa. 2 of chap. vi. which suited to the case before us, is



$$\cos \text{ZS} = \cos \text{PS} \cos \text{PZ} + \sin \text{PS} \sin \text{PZ} \cos \text{P}, \ldots (1^*)$$

or adopting the characters just specified

$$\cos n = \sin d \sin l + \cos d \cos l \cos \text{P} \ldots\ldots (1.)$$

Suppose $\text{P} = 0$, or the body in the meridian, then $\cos \text{P} = \text{rad} = 1$, and the fundamental equation becomes, $\cos \text{ZS} = \cos \text{PS} \cos \text{PZ} + \sin \text{PS} \sin \text{PZ} = \cos (\text{PS} - \text{PZ})$ whence $\text{ZS} = \pm (\text{PS} - \text{PZ}) = \pm \text{PS} \mp \text{PZ}$; $\text{PS} = \text{PZ} \pm \text{ZS}$;

$$\left.\begin{array}{l}\text{and PR or } 180^\circ - \text{PR} = \text{RS}'' + \text{PS}'' \text{ or HS}' + \text{PS}' \ldots \\ = \text{mer. alt. from north or south} + \text{declin} \ldots \\ = \tfrac{1}{2} \text{ sum. of mer. alts. in a circumpolar star} . \end{array}\right\} (2.)$$

7. If the horary angle $\text{P} = 90^\circ$, as it is when the body

is 6 hours from the meridian, then its cosine will vanish, and equa. 1, becomes

$$\cos n \text{ or } \sin a = \sin d \sin l \ldots\ldots\ldots (3.)$$

from which the altitude, a, becomes known.

In the same circumstances theorem 10, of right angled spherical triangles, gives

$$\cot \text{SZP} = \cot \text{PS} \sin \text{PZ} = \tan d \cos l \ldots. (4.)$$

by which the azimuth in that case may be determined.

8. When the body S is in the horizon, or S and V coincide, we have ZS $= 90°$, or $\cos n = 0$;

$$\text{hence } 0 = \sin d \sin l + \cos d \cos l \cos \text{P},$$

$$\text{or, } \cos \text{P} = -\frac{\sin d \sin l}{\cos d \cos l} = -\tan d \tan l \ldots (5.)$$

Here it is evident that when the declination and latitude are both of the same kind, cos P is negative, or P is greater than 90°; that is, the time occupied by the heavenly body in passing from the horizon to the meridian, or from the meridian to the horizon, exceeds 6 hours.

If the declination and latitude are of *different* kinds, cos P will be positive, and the time of passage from the horizon to the meridian, or from the meridian to the horizon, less than 6 hours. Thus, this theorem will serve to determine the times of the risings and settings of the sun, or other heavenly bodies, disregarding the changes of declination, and the effects of parallax and refraction.

To find the azimuth when S is in the horizon, we have from the principles of quadrantal triangles,

$$\cos \text{Z} = \frac{\cos \text{PS}}{\sin \text{PZ}} = \frac{\sin d}{\cos l} = \sin d \sec l \ldots. (6.)$$

9. If the body S be upon the prime vertical, then PZS $= 90°$, and the formulæ for right angled triangles give

$$\cos \text{P} = \tan \text{ZP} \cot \text{SP} = \cot l \tan d \ldots. (7.)$$

$$\cos \text{SZ} = \sin a = \frac{\cos \text{PS}}{\cos \text{PZ}} = \sin d \operatorname{cosec} l \ldots. (8.)$$

log tan d .. 16° 0′ 46″ ... 9·4578618
tan l .. 52° 12′ 35″ .... 10·1104697

The sum is log cos P .. 111° 43′ 21″ .... 9·5683315

Here since the latitude and declination are both of the same kind, cos P is negative, and belongs to an arc between 90° and 180°. This hour angle converted into time by dividing by 15 or multiplying by $\frac{4}{60}$, gives $7^h\ 26^m\ 53\frac{1}{3}^s$ from noon, for the time required.

In cases where great accuracy is needed, the change of declination in the interval must be regarded.

To find the azimuth, take the theorem

$$\cos z = \sin d \sec l.$$

Log sin d .. 16° 0′ 46″ .... 9·4406756
sec l .. 52° 12′ 35″ .... 10·2127004

The sum cos z .. 63° 14′ 44″ azimuth from the north .. 9·6533760

Example II.

Required the time of the sun's rising and setting at the equator, on the 14th of October, 1816.

It is not necessary in the solution of this example to look out the declination: for, since l the latitude is 0, tan l is 0, and consequently cos P = 0: therefore P = 90°, the hour angle of 6; that is, the sun rises and sets at 6 o'clock.

Cor. Hence it appears that at the equator all the heavenly bodies are 12 hours above and 12 hours below the horizon; neglecting the effects of refraction.

Example III.

Required the times when the sun is east or west, at

Cambridge, on May 4th, 1816, and his altitude at those times.

The theorems applicable to this example are 7 and 8, viz.

$$\cos P = \cot l \tan d, \text{ and } \sin a = \sin d \operatorname{cosec} l.$$

Thus, for the hour angle

Log cot *l* .. 52° 12′ 35″ .... 9·8895736
tan *d* .. 16° 0′ 46″ .... 9·4578618

The sum cos P .. 77° 8′ 27″ } .. 9·3474354
hour angle from noon

Then, for the altitude

Log sin *d* .. 16° 0′ 46″ .... 9·4406756
cosec *l* .. 52° 12′ 35″ .... 10·1022306

The sum sin *a* .. 20° 25′ 48″ .... 9·5429062

Note. By means of the equations to this example, the solar time when the sun is due east or west may be determined: and, therefore, in the summer half year, when the sun is *above* the horizon, when he is in the prime vertical, he will at those moments cast the shadows of opake objects in the east and west direction, and a horizontal perpendicular to such shadows will be a meridian line.

Example IV.

Given the latitude of the place 52° 12′ 35″ N. the sun's declination 15° 54′ 25″ N. and the sun's altitude 40°; to find his azimuth and the time from noon.

Here the time and azimuth may be found by means of equa. 12 and 13. Or, the azimuth being found by means of equa. 13, the time may be found by the proportion between the sines of the sides and opposite angles of spherical triangles, when we shall have

$$\sin P = \frac{\sin z \sin zs}{\sin sp} = \sin z \cos a \sec d.$$

The operation, which gives 119° 53′ 8″ for the azimuth from the north, and 2^h 55^m for the time from noon, is left for the learner's exercise.

Note. From such an example as this it will be easy to find the meridian, by simply drawing a line to make with the direction in which the sun is at the moment of observation, an angle equal to the azimuth found by this process. And this method, provided the sun's altitude be corrected for refraction and parallax, will be found very commodious in the determination of a meridian, as the changes in declination, which prevail in the method by corresponding altitudes, are in this avoided entirely. The same method serves evidently to determine the variation of the compass.

Example V.

In the latitude of 52° 12′ 35″ N. when the sun's declination is 15° 54′ 25″ N. what is the sun's azimuth at 9^h 5^m in the morning?

The azimuth, which is 119° 53′ 8″ from the north, may be readily found, as well as the angle at the sun and the zenith distance, by the equations marked 14 in the preceding investigation. The work is left for the pupil's exercise.

Note. An operation, like that required in this example, will serve to find the declination of any vertical plane, by knowing when the sun begins to shine upon it, or quits it.

Example VI.

At Peterborough, in latitude 52° 32′ N., on February 21, 1801, at what time from that star's being on the meridian, was the apparent motion of *Dubhe* or *α Ursa majoris* vertical; and what were the altitude and azimuth of the star at that time?

On the year and day specified, the declination, *d*, of the star was 62° 50′. The logarithmic opperation corresponding to equa. 9, viz.

$$\cos P = \cot d \tan l,$$

will give 47° 57′ 46″, equivalent to $3^h\,11^m\,51^s$, in time, for the hour angle from the meridian.

Also, equa. 10, or $\sin z = \cos d \sec l$,
will give 48° 38′ 28″ azimuth from the north:

And equa. 11, or $\sin a = \sin l \operatorname{cosec} d$,
will give 63° 8′ 29″, for the altitude of the star.

The operation is again left for the learner's exercise.

Note. Allied to the principle of this example is the method of finding, not merely the meridian, but the latitude, simply by observing two azimuths of the same known fixed star with the *azimuth compass*. If the azimuth of a circumpolar star be taken on each side of the meridian when at the greatest from the elevated pole, that is, when the apparent motion of the star is vertical; then it is evident that *half the sum* of these two azimuths will be the true azimuth from the north or south, according as the latitude of the place is north or south, and that *half the difference* will be the variation of the needle. Also, recurring to the diagram at page 158, there are given in the triangle PZ*s* right angled at *s*, the star's co-declination P*s*, and the azimuth PZ*s*, whence the co-latitude ZP becomes known by means of Case 4 of right angled triangles; from which we have

$$\cos l = \frac{\cos d}{\sin z} = \cos d \operatorname{cosec} z$$

Example. VII.

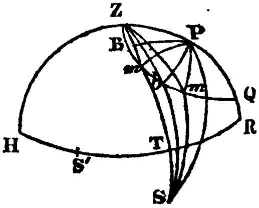
At what time on the 9th of August, 1817, will the twilight begin and end at Cambridge?

Here the sun's declination for the proposed time is 15° 55′ 25″ N. and the latitude 52° 12′ 35″ N. Putting 108° for ZS, 37° 47′ 25″ for PZ, and 74° 4′ 35″ for PS, in equa. 12, the resulting hour angle reduced to time will give $10^h\,12\frac{1}{2}^m$ for the time from noon when twilight begins and ends.

PROBLEM III.

13. Given the latitude of a place to find the time and the duration of shortest twilight.

Let P be the elevated pole of the world, Z the zenith of the observer, S the sun 18° below the horizon, ZS $= 90° + 18°$, or more generally ZS $= 90° + 2a$. The observer, whose zenith is Z, will then see the commencement of the morning twilight.



By reason of the diurnal rotation, the declination circle PS turning about the axis will bring the sun from S to S′ in the horizon, if it be the heavens which turn about the earth: on the contrary hypothesis of the earth turning on its axis, the zenith Z will approach nearer to S by describing about P the little circle Z$mb$Q, such that the distances PZ, Pm, Pb, PQ, are all equal.

The same consequences as to this problem may be drawn from both hypotheses; but the latter is a little the most convenient.

When, therefore, the zenith shall have descended from Z to any point m, such that mS $= 90°$ the sun will appear at 90° from the zenith, the day will commence and the twilight terminate; and the arc Zm of the little circle will be the measure of the angle ZPm, and consequently of the duration of the twilight. To determine this angle draw the arc ZBm of a great circle, and to its middle B the perpendicular arc PB; then will

$$\sin \tfrac{1}{2}\,\mathrm{ZP}m = \sin \mathrm{ZPB} = \frac{\sin \frac{1}{2}\,\mathrm{Z}m}{\sin \mathrm{PZ}} = \frac{\sin \frac{1}{2}\,\mathrm{Z}m}{\cos \mathrm{L}}.$$

Now, in the spherical triangle ZmS, Sm + mZ > ZS, or $90° + m\mathrm{Z} > 90° + 2a$. Therefore $m\mathrm{Z} > 2a$, and $\frac{1}{2}m\mathrm{Z} > a$. Let $\frac{1}{2}m\mathrm{Z} = a + x$: then

$$\sin \tfrac{1}{2}\mathrm{ZP}m = \frac{(\sin a + x)}{\cos \mathrm{L}} = \frac{\sin a \cos x + \cos a \sin x}{\cos \mathrm{L}}$$

$$= \frac{\sin a + \cos a \sin x - 2 \sin a \sin^2 \tfrac{1}{2}x}{\cos \text{L}}$$

[because $1 - \cos x = 2 \sin^2 \tfrac{1}{2}x$. ch. iv. art. 23.]

$$= \frac{\sin a}{\cos \text{L}} + \frac{2 \sin \tfrac{1}{2}x (\cos a \cos \tfrac{1}{2}x - \sin a \sin \tfrac{1}{2}x.)}{\cos \text{L}}$$

$$= \frac{\sin a}{\cos \text{L}} + \frac{2 \sin \tfrac{1}{2}x \cos (a + \tfrac{1}{2}x)}{\cos \text{L}}.$$

Now $\tfrac{1}{2}x$ is essentially positive, and a and $\tfrac{1}{2}x$ small angles such that $a + \tfrac{1}{2}x < 90°$;

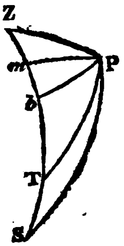
$$\text{therefore } \sin \tfrac{1}{2}\text{ZP}m > \frac{\sin a}{\cos \text{L}}.$$

It is farther evident that the twilight will be longer as x is greater, and shorter as x is less; and that it will be the shortest possible if x can be nothing; because in that case the second term of the second member of the equation will vanish. But this is what occurs when the triangle ZmS is reduced to the arc ZS, that is, when the point m falls on b, or, when the distance PS is such that the part Zb of the vertical ZS lying within the little circle ZmQ is equal to $2a$, and the exterior part bS equal to 90°. It is also manifest that if PS increases the opposite angle PZS will become enlarged, and that on the contrary the said angle will diminish if PS diminishes. In these variations the point m will approach to or recede from Z, the intercepted part Zb will vary, and it *may* vary between the limited 0 and ZQ = 2PZ. Thus the intercepted part may have all values between 0 and 2(90° − L) = 180° − 2L, and consequently may have the value $2a$. Hence, in the case where Zb = $2a$, and bS = 90°, the shortest twilight will obtain; and its semiduration in degrees will be found by the equation.

$$\sin \text{ZPB} = \frac{\sin a}{\cos \text{L}} = \sin a \sec \text{L} \; .. \; (1.)$$

$$\text{duration in time} = \tfrac{2}{15} \text{ ZPB} \; .... \; (2.)$$

14. Other theorems may be deduced from the resulting



construction. On the arc $zb = 2a$ and with the complement ZP of the latitude, constitute the isosceles triangle Z$\mathrm{P}b$, and let fall the perpendicular $\mathrm{P}m$; Z$\mathrm{P}b$ will be the angle which measures the duration. Prolong zb till $bs = 90°$, and draw the arc PS, which will be the sun's polar distance for the day of the shortest twilight.

Now, $\cos m\mathrm{P} = \dfrac{\cos \mathrm{PZ}}{\cos \mathrm{Z}m} = \dfrac{\cos \mathrm{PS}}{\cos m\mathrm{S}}$ (chap. vi. 27.)

$$\text{therefore } \cos \mathrm{Z}m : \cos m\mathrm{S} :: \cos \mathrm{PZ} : \cos \mathrm{PS};$$
$$\text{or, } \cos a : \cos (90° + a) :: \sin \mathrm{L} : \sin \mathrm{D};$$
$$\text{or, } \cos a : -\sin a :: \sin \mathrm{L} : \sin \mathrm{D};$$
$$\text{whence } \sin \mathrm{D} = -\frac{\sin a}{\cos a} \sin \mathrm{L} = -\tan a \sin \mathrm{L} \quad (3.)$$

15. Farther, the right angled triangle $\mathrm{PZ}m$, gives $\tan \mathrm{Z}m = \cos \mathrm{Z} \tan \mathrm{PZ}$ (chap. vi. 26.)

$$\text{and } \cos \mathrm{Z} = \tan \mathrm{Z}m \cot \mathrm{PZ} = \tan a \tan \mathrm{L} \quad (4.)$$

Now Z is the sun's azimuth at the commencement of the twilight, and $\mathrm{PZ}b = \mathrm{P}b\mathrm{Z} = 180° - \mathrm{P}b\mathrm{S}$. But $\mathrm{P}b\mathrm{S}$ is the sun's azimuth at the instant when his centre is at the true horizon: therefore *the sun's azimuths at the beginning and end of the crepusculum are supplements to each other on the day of the shortest twilight.*

Hence, since $\cos \mathrm{PZS} = \tan a \tan \mathrm{L}$

$$\text{we have } \cos \mathrm{P}b\mathrm{S} = -\tan a \tan \mathrm{L} \;..\; (5.)$$

16. ZPS and bPS are the hour angles at the beginning and end of the twilight; let the former be denoted by P$'$, the latter by P: then $\mathrm{ZPS} - b\mathrm{PS} = \mathrm{P}' - \mathrm{P} = \mathrm{ZP}b$, the angle which measures the duration of twilight. Hence we have from what has been done above,

$$\sin \tfrac{1}{2}(\mathrm{P}' - \mathrm{P}) = \frac{\sin a}{\cos \mathrm{L}} \cdots\cdots (6.)$$

Also $m\mathrm{PS} = b\mathrm{PS} + m\mathrm{P}b = \mathrm{P} + \frac{1}{2}(\mathrm{P}' - \mathrm{P}) = \frac{1}{2}(\mathrm{P}' + \mathrm{P})$

$$\text{and } \sin m\mathrm{PS} = \frac{\sin \mathrm{S}m}{\sin \mathrm{PS}} = \frac{\sin (90° + a)}{\cos \mathrm{D}} = \frac{\cos a}{\cos \mathrm{D}};$$

$$\text{that is, } \sin \tfrac{1}{2}(\mathrm{P}' + \mathrm{P}) = \frac{\cos a}{\cos \mathrm{D}} \ldots\ldots (7.)$$

Hence from equa. 6 and 7, we may determine P′ and P, the hour angles at the beginning and end of the twilight, D the sun's declination being determined by equa. 3.

$$\text{We have also } \cos \mathrm{S} = \frac{\sin \mathrm{L}}{\cos \mathrm{D}} \ldots\ldots (8.)$$

17. Let $\mathrm{ST} = 2a = zb$; and draw PT, then is $\mathrm{ZT} = 90°$, and T is a point in the horizon for the moment when the zenith was in Z. The triangle PZT gives (chap. vi. art. 21.)

$$\cos \mathrm{PT} = \cos \mathrm{Z} \sin \mathrm{PZ} \sin \mathrm{ZT} + \cos \mathrm{PZ} \cos \mathrm{ZT} = \cos \mathrm{Z} \sin \mathrm{PZ}$$
$$= \tan a \tan \mathrm{L} \cos \mathrm{L} = \tan a \sin \mathrm{L} = \sin \mathrm{D}.$$

Therefore $90° - \mathrm{PT} = \mathrm{D}$, or $\mathrm{PT} = 90° - \mathrm{D}$.

But $\mathrm{PS} = 90° + \mathrm{D}$; therefore $\mathrm{PT} + \mathrm{PS} = 180°$ ∴ (9.)

This is another remarkable symptom of the shortest twilight.*

Example.

Required the day on which the twilight is shortest, at Woolwich, N. lat. 51° 28′ 40″, in the year 1816, with its duration and the time of its beginning and end.

First, for the declination, $\sin \mathrm{D} = - \tan a \sin \mathrm{L}$

To log tan a .. 9° .... 9·1997125
Add log sin L .. 51° 28⅔′ .... 9·8934104

The sum, log sin D. .. − 7° 7′ 5″ .. 9·0931229

Thus the declination is south, and answers to March 2d, and October 11th.

---

* Many other curious particulars connected with this problem are given in *Astronomie Théorique et Pratique*, par M. Delambre, chap. 14; whence the above investigation has been extracted. Another elegant investigation may be seen in *Connaissance des Tems*, pour l'an 1818, pp. 382—404.

Next, for the duration, $\sin \frac{1}{2}(P' - P) = \frac{\sin a}{\cos L}$

From log sin a .. 9° .... 9·1943324
Take log cos L .. 51° 28⅔′ .... 9·7943613

Rem. log sin.... 14° 32′ 41″.. 9·3999711

Hence $P' - P = 29° 5' 22''$, or in time $1^h\, 56^m\, 21\frac{1}{2}^s$.

Farther, $\sin \frac{1}{2}(P' + P) = \frac{\cos a}{\cos D}$.

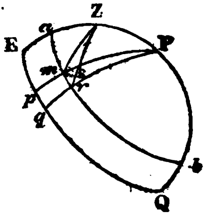
Hence, from log cos a .. 9° .... 9·9946199
take log cos D .. 7° 7′ 5″.... 9·9966399

Rem. log sin.... 95° 31′ 19″.... 9·9979800

This value of $\frac{1}{2}(P' + P)$ reduced to time is equivalent to $6^h\, 22^m\, 5\frac{1}{2}^s$, while $\frac{1}{2}(P' - P)$ is $58^m\, 10\frac{3}{4}^s$. The sum of these is $P' = 7^h\, 20^m\, 16\frac{1}{4}^s$, the end of the evening twilight; their difference is $P = 5^h\, 23^m\, 54\frac{3}{4}^s$, the time of the sun setting or the beginning of evening twilight. These respectively taken from 12, leave the times of the beginning and end of the morning twilight.

Problem IV.

18. Assuming the error in taking an altitude, it is required to determine the corresponding error in time, in the case of example 4, prob. ii.



Let EQ in the marginal figure be the equator, P the elevated pole, ab the parallel of declination in which the sun or other heavenly body is on the day of observation; and let r be the real and s the apparent place of the body, or rs the error in altitude. Draw ms parallel to the horizon, from Z the zenith draw zm, zr, and from P the pole, the meridians, Pmp, Prq, to pass through m, r, and cut the equator in the points p, q. Then m and r will be the apparent and real places of the body on the parallel of declination, and the arc mr on that circle, or pq on the equator, will measure the angle $m\text{P}r$, the corresponding error in time. The triangle msr being, of course, exceedingly small, may be regarded as a rectilinear triangle right angled at s.

Hence, $sr : rm :: \sin smr : \text{rad}.$

and $mr : pq :: \cos qr : \text{rad}.$

Conseq. multiplying the corresponding terms we have

$$sr : pq :: \sin smr \cos qr : \text{rad}^2$$

$$\text{Whence } pq = sr \frac{\text{rad}^2}{\sin smr \cos qr} \ldots. (1.)$$

But $\text{Z}r\text{P} = smr$, srm being the comp. of each;

and $\sin \text{Z}r\text{P}$ $(= \sin smr) : \sin \text{PZ} :: \sin r\text{ZP} : \sin \text{P}r$;

Conseq. $\sin smr \sin \text{P}r = \sin smr \cos qr = \sin \text{PZ} \sin r\text{ZP}$.

Therefore, by substituting for the denominator in equa. 1, its equivalent in the last, we have

$$pq = \frac{rs\,.\,\text{rad}^2}{\sin \text{PZ} \sin r\text{ZP}} = \frac{rs}{\cos \text{L} \sin \text{Z}} \ldots. (2.)$$

Cor. Hence, since sin azimuth is a *maximum* when the body is on the prime vertical, the error in time will then be a *minimum*, the refraction being accurately allowed for. Wherefore,

Cor. 2. It is best to deduce the time from an altitude taken when the sun or other body is on or near the prime vertical.

[See, on kindred topics, chap. xi.]

Example.

Suppose that in example 4, prob. 2. the error in altitude was 1′, what would be the corresponding error in time?

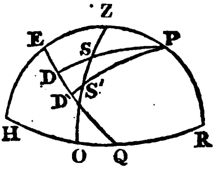
$$pq = \frac{1}{\cos 52^\circ 12' 35'' \sin 119^\circ 53' 8''} = \frac{1}{\cdot 6128 \times \cdot 8670} = 1^\circ.882$$

This in time is $= 7{\cdot}528$ seconds.

PROBLEM V.

19. The right ascensions and declinations of two fixed stars, observed on the same vertical given in position, together with their distance, or the arc of that vertical included between them, being known; it is required to determine the latitude of the place of observation.

If s, s', in the annexed diagram, be the places of the two stars in the vertical ZSO, HR the horizon, P the elevated pole, and EQ the equator; then it is evident that the three sides of the triangle PSS' are given, as well as the angle at P. For PS, PS', are the codeclinations of the stars (known by hypothesis), SS' their distance on the vertical given in position, and the angle SPS' the measure of the difference of their right-ascensions. Hence, by the proportionality of the sines of opposite sides and angles, we shall have,



$$\sin SS' : \sin SPS :: \sin S'P : \sin PSS' = \sin PSZ$$

$$= \frac{\sin S'P}{\sin SS'} \sin SPS'.$$

Thus, two angles and a side opposite to one of them, will become known in the triangle PZS, namely, the azimuth, PZS, the angle PSZ, and the side PS; whence we shall have

$$\sin PZS : \sin PSZ :: \sin PS : \sin PZ = \cos L =$$

$$\frac{\sin PS \sin PSZ}{\sin PZS} = \frac{\sin PS \sin PS' \sin SPS'}{\sin SS' \sin PZS}$$

$$= \sin PS \sin PS' \sin SPS' \operatorname{cosec} SS' \operatorname{cosec} PZS.$$

20. *Cor.* 1. It is evident that the hour angle ZPS may be readily found from the same data; and therefore if the sun's place and his right ascension be known, the difference between it and that of the star, s, will become

known; and if from this difference the arc ED be taken, the remainder will give the arc of the equator contained betwixt the meridian of the place and that which passes through the sun, whence the moment of observation may at once be deduced. It is equally obvious that, by pursuing the computation, the *altitude* of each of the stars at the moment of observation may be found.

21. *Cor.* 2. If, instead of the azimuth of the two stars, their altitudes were given, we might readily determine the elevation of the pole; for, in that case we should know the three sides of the triangle PSS′ with the angle P; and thence the angle PSZ between the given sides PS, ZS, of the triangle PSZ.

22. *Cor.* 3. So again, if besides the declinations and right ascensions of the stars observed on the same vertical, the hour angle ZPS were given, the latitude might be ascertained; for then, in the triangle PSZ, the angles ZPS, ZSP, and the side SP, would be known, and ZP would be determinable by case 3, prob. 3, of oblique spherical triangles.

23. *Cor.* 4. Hence, from this problem and corollaries the latitude of a place may be determined with great facility; a plumb line serving to show when two stars are in the same vertical, and the altitude being susceptible of being taken with sufficient accuracy for common purposes without employing large instruments. But correct tables of the places of the stars are indispensable.

Example.

Two stars, the right ascension of one of which S is 78° 24′, and its declination 27° 25′, the right ascension of the other, S′, is 104° 52′, and its declination 12° 18′, of the same kind with the former, are distant from each other 28° 30′, on a vertical whose azimuth EZO is 73° 36′. Required the latitude of the place?

Here PS = 90° − 27° 25′ = 62° 35′, PS′ = 90° − 12° 18′ = 77° 42′,

ss′ = 28° 30′, sps′ = 104° 52′ − 78° 24′ = 26° 28′, and
pzs = 180° − 73° 36′ = 106° 24′.

The logarithmic computation from the formula is

Log sin ps	62° 35′	9·9482572
sin ps′	77° 42′	9·9899148
sin sps′	26° 28′	9·6490203
cosec ss′	28° 30′	10·3213371
cosec pzs	106° 24′	10·0180392
Log cos L	32° 23′ 18″	9·9265686

Hence the latitude is 32° 23′ 18″, of the same kind with the declinations of the stars.

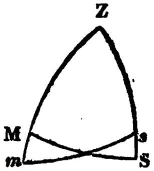
Problem VI.

24. Given the apparent altitudes of the moon and sun, or of the moon and a fixed star, and their apparent distance to determine their true distance.

This is an important problem in navigation, being of essential use in one of the best methods of determining the longitude at sea. The true distance of the moon and the sun, or a star at any given instant, being ascertained, we are enabled by means of the "Nautical Almanac," and the Ephemerides of different countries, to find the time at the first meridian, which corresponds with the moment of observation: the difference between these times, reduced to degrees at the rate of 15 to an hour, shows the longitude required.

Besides the error arising from the imperfection of instruments, the observed altitudes of the heavenly bodies are affected by three causes, the *depression of the horizon*, the *refraction* of rays of light in passing through the air, and the *parallax*, or *the inclination of two visual rays passing from the celestial body, one to the earth's centre, the other to the point where the observer is placed on its surface.* The principal works on nautical astronomy contain tables by which the requisite allowances may be made for these at any place and for any heavenly body: and the purpose of making such allowances is, to reduce the observed or apparent altitude of any body at or above the surface of the earth, to the real altitude at which it would appear from the earth's centre if light were transmitted in right lines.

25. The apparent altitude of the sun always surpasses the true; for the refraction elevates the sun more than the parallax depresses that luminary. With regard to a fixed star the parallax is evanescent. Hence, the true place of the sun, or of a fixed star, will always be *below* the apparent place. The moon, on the contrary, is more depressed by parallax than it is elevated by refraction. Hence the true place of the moon will be always *above* its apparent place.



26. Let, then, s be the apparent place of the sun or star, S its true place, on the same vertical zs, m the apparent place of the moon, M its true place, on the same vertical zm, ms the apparent distance of the two bodies. Then, there are known $H = 90 - zs$, apparent altitude of the sun or star; $H' = 90° - zm$, apparent altitude of the moon; $D = ms$, their apparent distance: also, $h = H - r + p = 90° - ZS$, true altitude of the sun or star; and $h' = H' - r' + p' = 90° - ZM$, true altitude of the moon; r and p, r' and p', being the refraction and parallax corresponding respectively to the apparent altitude of the bodies. Consequently, in the spherical triangle zms are given all the sides to find the angle Z: and then in the triangle ZMS are given two sides ZM, ZS, and their included angle Z, to find the third side $MS = d$ the true distance required. The solution of the problem thus conducted is obviously as simple and natural as can well be wished, to men acquainted with theory and accustomed to computation: yet it has been found embarrassing to mariners; on which account most writers on the subject of navigation have investigated other

rules, several of which (especially those which depend upon subsidiary tables) are more direct and expeditious in operation than the original rules deduced without modification from the two triangles *zms*, ZMS.

27. The second general theorem of spherical trigonometry, when applied to these two triangles, gives

$$\cos Z = \frac{\cos sm - \cos zm \cos zs}{\sin zm \sin zs} = \frac{\cos SM - \cos ZM \cos ZS}{\sin ZM \sin ZS}.$$

Or, employing the preceding notation,

$$\cos Z = \frac{\cos D - \sin H \sin H'}{\cos H \cos H'} = \frac{\cos d - \sin h \sin h'}{\cos h \cos h'}.$$

But since $\cos (H + H') = \cos H \cos H' - \sin H \sin H'$, we have $\sin H \sin H' = \cos H \cos H' - \cos (H + H')$, and, in like manner

$$\sin h \sin h = \cos h \cos h - \cos (h + h');$$

$$\text{whence, } \cos Z = \frac{\cos D + \cos (H + H') - \cos H \cos H'}{\cos H \cos H'},$$

$$= \frac{\cos d + \cos (h + h') \cos h \cos h'}{\cos h \cos h}.$$

Taking from the second member of this equation $\frac{\cos H \cos H'}{\cos H \cos H'} = 1$, and from the third $\frac{\cos h \cos h'}{\cos h \cos h'} = 1$; we have for the result

$$\frac{\cos D + \cos (H + H')}{\cos H \cos H'} = \frac{\cos d + \cos (h + h')}{\cos h \cos h'} \ldots \text{(A)}$$

Now it is evident from the formulæ in chap. iv. that $\cos D + \cos (H + H') = 2 \cos \tfrac{1}{2} (H + H' + D) \times$
$\cos \tfrac{1}{2} (H + H' - D)$,
$\cos (h + h') = 2 \cos^2 (h + h') - 1$,
$\cos d = 1 - 2 \sin^2 \tfrac{1}{2} d$.

These values being respectively substituted for the several quantities in equa. A, it will be transformed into

$$\frac{2 \cos \tfrac{1}{2} (H + H' - D) \cos \tfrac{1}{2} (H + H' + D)}{\cos H \cos H'}$$

$$= \frac{1 - 2 \sin^2 \tfrac{1}{2} d + 2 \cos^2 \tfrac{1}{2} (h + h') - 1}{\cos h \cos h'}.$$

Multiplying by cos h cos h', and reducing, we have

$$\frac{\cos h \cos h'}{\cos \mathrm{H} \cos \mathrm{H}'} \cos \left(\frac{\mathrm{D} + \mathrm{H} + \mathrm{H}'}{2} - \mathrm{D}\right) \cos \frac{\mathrm{D} + \mathrm{H} + \mathrm{H}'}{2}$$

$$= -\sin^2 \tfrac{1}{2}d + \cos^2 \tfrac{1}{2}(h + h') \ldots\ldots (\mathrm{B})$$

This, if S be put $= \frac{1}{2}(\mathrm{D} + \mathrm{H} + \mathrm{H}')$ becomes,

$$\cos \mathrm{S} \cos (\mathrm{S} - \mathrm{D}) \frac{\cos h \cos h'}{\cos \mathrm{H} \cos \mathrm{H}'} = -\sin^2 \tfrac{1}{2}d + \cos^2 \tfrac{1}{2}(h + h').$$

Hence

$$\sin^2 \tfrac{1}{2}d = \cos^2 \tfrac{1}{2}(h + h') - \frac{\cos \mathrm{S} \cos (\mathrm{S} - \mathrm{D}) \cos h \cos h'}{\cos \mathrm{H} \cos \mathrm{H}'}$$

$$= \cos^2 \tfrac{1}{2}(h + h)\left(1 - \frac{\cos \mathrm{S} \cos (\mathrm{S} - \mathrm{D}) \cos h \cos h'}{\cos \mathrm{H} \cos \mathrm{H}' \cos^2 \frac{1}{2}(h + h')}\right).$$

$$\text{Put } \sin^2 \mathrm{F} = \frac{\cos \mathrm{S} \cos (\mathrm{S} - \mathrm{D}) \cos h \cos h' \sec \mathrm{H} \sec \mathrm{H}'}{\cos^2 \frac{1}{2}(h + h')} \ldots (\mathrm{C})$$

Then the preceding equation becomes $\sin^2 \frac{1}{2}d$
$= \cos^2 \frac{1}{2}(h + h')(1 - \sin^2 \mathrm{F}) = \cos^2 \frac{1}{2}(h + h') \cos^2 \mathrm{F}$,
whence $\sin \frac{1}{2}d = \cos \frac{1}{2}(h + h') \cos \mathrm{F} \ldots\ldots (\mathrm{D})$

This is the formula of *Borda*: it has the advantages of requiring only the usual tables; and of being free from ambiguity.

28. The logarithmic formulæ deduced from equations C and D, are

$$2 \log \sin \mathrm{F} = \log \cos \mathrm{S} + \log \cos (\mathrm{S} - \mathrm{D}) + \log \cos h$$
$$+ \log \cos h' + \log \sec \mathrm{H} + \log \sec \mathrm{H}'$$
$$+ 2 \log \sec \tfrac{1}{2}(h + h') - 60 \ldots\ldots (\mathrm{E})$$

$$\log \sin \tfrac{1}{2}d = \log \cos \tfrac{1}{2}(h + h) + \log \cos \mathrm{F} - 10 \ldots (\mathrm{F})$$

The secants being introduced into the numerator for the cosines in the denominator, render the logarithmic computation entirely additive.

Example.

The apparent distance of the moon's centre from the star Regulus being 63° 35′ 13″, when the apparent altitude of the moon's centre was 28° 29′ 44″, the apparent altitude of the star 45° 9′ 12;″ the moon's correction, or the difference between the refraction and the parallax in altitude $48' \, 1''$; the star's correction, or the refraction $57''$. Required the true distance?

1st *Method*, directly from the triangles *zms*, ZMS.

In the triangle *zms* (last figure) are given

$$zm = 90° - 28° \, 29' \, 44'' = 61° \, 30' \, 16'', \; zs = 90° - 45° \, 9' \, 12'' = 44° \, 50' \, 48''$$

$$sm = 63° \, 35' \, 13'' :$$ to find the angle z.

This may be effected by ch. vi. art. 38, equa. 2. which, suited to the present case, becomes

$$\tan \tfrac{1}{2}z = \sqrt{\frac{\sin \frac{1}{2}(zm + ms - sz) \sin \frac{1}{2}(ms + zs - zm)}{\sin \frac{1}{2}(zm + zs - ms) \sin \frac{1}{2}(ms + zs + zm)}} :$$

and this, when performing the logarithmic computation, may be best accomplished by *adding* the log cosecants of the terms in the denominator, instead of *substracting* their log sines, and deducting 10 from the index, at last.

Thus, $\sin \frac{1}{2}(zm + ms - sz)$ .. $40° \, 7' \, 20''$.. 9.8091692
$\sin \frac{1}{2}(ms + zs - zm)$ .. $23° \, 27' \, 53''$.. 9·6000842
$\text{cosec} \frac{1}{2}(zm + zs - ms)$ .. $21° \, 22' \, 38''$.. 10·4382946
$\text{cosec} \frac{1}{2}(ms + zs + zm)$ .. $84° \, 58' \, 8''$.. 10·0016765

2 | 39·8492245

The half sum − 10 is $\tan \frac{1}{2}z$ .. $40° \, 3' \, 7''$ .. 9·9246122

Consequently $z = 80° \, 6' \, 14.''$

Then there are given, in the triangle ZMS,

$$ZM = 90° - (28° \, 29' \, 44'' + 48' \, 1'') = 60° \, 42' \, 15''$$
$$ZS = 90° - (45° \, 9' \, 12'' - 57') = 44° \, 51' \, 45''$$

and the included angle $z = 80° \, 6' \, 14''$.

The third side may be obtained as in case 2, prob. 2, by finding a subsidiary angle φ such that

$$\tan \varphi = \cos 80° \, 6' \, 14'' \tan 44° \, 51' \, 45''$$

In logs ...... $\cos 80° \, 6' \, 14''$ .... 9·2351804
$\tan 44° \, 51' \, 45''$ .... 9·9979155

The sum $\tan \varphi = \tan \; 9° \, 42' \, 21''$ .... 9·2330959

$\cos \mathrm{MS} = 44° 51′ 45″ \cos (60° 42′ 15″ - \varphi) \sec \varphi.$

In logs…… cos 44° 51′ 45″ …. 9·8506086
cos 50° 59′ 54″ …. 9·7988874
sec 9° 42′ 21″ …. 10·0062613

The sum cos MS = cos 63° 5′ 11″ …. 9·6557573

Thus the true distance between the moon and star is found to be 63° 5′ 11″.

2dly. *By Borda's Theorem.*

Here we have H = 45° 9′ 12″, h = 45° 8′ 15″, H′ = 28° 29′ 44″, h' = 29° 17′ 45″, $h + h'$ = 74° 26′, S = 68° 37′ 4½″, S − D = 5° 1′ 51½″

The logarithmic computation for F, therefore, is

cos S …… 68° 37′ 4½″ …. 9·5617995
cos (S − D).. 5° 1′ 51½″ …. 9·9983236
cos h …… 45° 8′ 15″ …. 9·8484403
cos h' …… 29° 17′ 45″ …. 9·9405687
sec H …… 45° 9′ 12″ …. 10·1516804
sec H′ …… 28° 29′ 44″ …. 10·0560832
2sec ½($h + h'$) 37° 13′ 0″ …. 20·1977876

The sum − 60 = 2 sin F………. 19·7546833

Its half is sin F …. 48° 56′ 1″ …. 9·8773416

Then to find ½d, and thence d, we have

cos F………. 48° 56′ 1″ …. 9·8175211
cos ½ ($h + h'$)..37° 13′ 0″ …. 9·9011062

The sum is sin ½d……31° 32′ 34″ …. 9·7186273

Consequently d is 63° 5′ 8″.

Note. Here the difference in the results is only 3 seconds; which is quite as little as can be expected when tables are employed in which the computer has to proportion for the seconds. Had the value of F been taken

48° 56′ instead of 48° 56′ 1″ (and it is evident from the common tables that it lies between the two), then *d* would have been found about 63° 5′ 15″. The mean between these two results is 63° 5′ 11½″.

Example II.

Given the apparent distance between the centres of the sun and moon 83° 57′ 33″, the apparent altitude of the sun's centre 48° 27′ 32″, its true altitude 48° 26′ 49″, the apparent altitude of the moon's centre 27° 34′ 5″, its true altitude 28° 20′ 48″. Required the true distance of the centres of the two luminaries.

Ans. 83° 20′ 55″.

Example III.

The apparent distance of the moon's centre and a star was 2° 20′, when the apparent altitude of the star's centre was 11° 14′, and that of the moon's 9° 39′; the moon's correction was 51′ 30″, the star's 4′ 40″. Required the true distance of their centres.

Ans. 1° 49′.

Note 2. In examples of this kind the sum of the apparent zenith distances must always be greater than the apparent distance; for if not they cannot form a spherical triangle. Thus it will be found that example 4, pa. 31, of the Requisite Tables, 2d edition, relates to an impossible case. The third side 103° 29′ 27″ is greater than 89° 50′ 22″ the sum of the other two.

Note 3. This being an important problem in nautical astronomy, we shall here refer to other works where more compendious rules, founded principally upon subsidiary tables, are given. Such are, the *Requisite Tables* to be used with the Nautical Almanac; *Mackay on the Longitude; Mendoza's Tables for Navigation and Nautical Astronomy; Myers's Translation of Rossel on the Longitude; Andrew's Astronomical and Nautical Tables; Kelly's Spherics.* See also, Mr. *Sanderson's* rules in the Ladies' Diary for 1787, or in *Leybourn's* collection of the Diaries, vol. iii. Dr. *Brinkley's* in the Irish Transactions for 1808, and various others in *Delambre's Astronomy*, vol. iii. chap. 36.

PROBLEM VII.

29. Given the longitudes and latitudes of two places upon the earth, regarded as a globe, to find their itinerary distance, that is, the arc of the great circle comprehended between them.

Let A and B be the two places on the surface of the terraqueous globe, of which P is one of the poles, and conceive a spherical triangle PAB to be described, such that PA, PB, shall be the respective distances of the two places from the pole P, or their respective co-latitudes, and the angle APB the difference of longitude of those two places. Hence, L and L′ being the respective latitudes, and P the difference of longitude, there are given in the triangle PAB, two sides, viz. $PA = 90° \mp L$, $PB = 90° \mp L'$, and P, to find the third side AB. The problem, therefore, belongs to case 2, prob. 2, of oblique spherical triangles, the appropriate formulæ for which, suited to the present case, become

$$\tan \varphi = \cos P \tan PB \ldots . (1.)$$

for the subsidiary angle:

$$\text{and } \cos AB = \cos PB \sec \varphi \cos (PA - \varphi) \ldots . (2.)$$

Example I.

Given the latitude of the observatory of Paris, 48° 50′ 14″ N. that of the observatory of Pekin, 39° 54′ 13″ N. and their difference of longitude 114° 7′ 30″: to find their distance.

Here since the latitudes are both of the same kind, we have

$$PA = 90° - L = 90° - 48° 50' 14'' = 41° 9' 46''$$
$$PB = 90° - L' = 90° - 39° 54' 13'' = 50° 5' 47''$$
$$\text{and } P = \text{diff. of long.} = 114° 7' 30''.$$

Hence to log cos P .... 114° 7′ 30″.. 9·6114352
add tan PB .... 50° 5′ 47″.. 10·0776707

The sum − 10 = log tan φ − 26° 2′ 53″.. 9·6891059

Now, since P exceeds 90° its cosine is negative, and consequently φ must be taken negatively. Hence PA − φ = 41° 9′ 46″ + 26° 2′ 53″ = 67° 12′ 39″, and the work indicated by equa. (2) will stand thus:

Log cos PB ...... 50° 5′ 47″.... 9·8071953
cos (PA − φ) 67° 12′ 39″.... 9·5880938
sec φ........ 26° 2′ 53″.... 10·0465177

The sum − 20, cos AB 73° 56′ 40″.... 9·4418068

Thus the distance required is 73° 56′ 40″ of a great circle, or in English miles, reckoning 69½ to a degree, it is 5139 nearly.

Example II.

The latitude of St. Helena is 15° 55′ S. its longitude 5° 49′ W.; the latitude of the Bermudas 32° 35′ N. longitude 63° 32′ W. Required their distance?

Here since the latitudes are of different kinds, we shall have

$$PA = 90° + L = 90° + 15° 55' = 105° 55'.$$
$$PB = 90° - L' = 90° - 32° 35' = 57° 25'$$
$$\text{and } P = 63° 32' - 5° 49' = 57° 43'.$$

The preceding formulæ applied to these data, will give 73° 26′ for the distance required.

Example III.

What is the distance on a great circle between St. Mary's, latitude 37° N., longitude 25° W., and Cape Henry on the same parallel of latitude, but in longitude 76° 23′ W.?

Ans. 40° 31, which is about 36 miles less than the distance measured upon the parallel itself.

Example IV.

Required the distance on a great circle between the island of St. Thomas, latitude 0°, longitude 1° E., and Port St. Julian, in latitude 48° 51′ S., and longitude 65° 10′ W.

Ans. 74° 35′.

Note. In the 3d example the operation becomes shortened because the triangle PAB is *isosceles;* in the 4th it is shortened because the triangle is *quadrantal.*

PROBLEM VIII.

30. To determine the elongation of a planet from the sun, at the time when it appears stationary; on the supposition of concentric circular orbits, in one and the same plane.

As thus restricted the problem may be solved by means of the principles of plane trigonometry, combined with those of central forces in dynamics.

Let two concentric circles E*e*I, P*p*Q, be drawn, the former to represent the orbit of the earth, the latter that of the planet, the common centre S of both circles being imagined the place of the sun. While the earth moves from E to *e* in its orbit, let the planet move from P to *p* in its orbit; then, when the planet appears *stationary* the right lines EP, *ep*, drawn from the earth to the planet will be parallel. In that state of things draw the radii SE, S*e*, SP, S*p*; from E and P draw tangents to their respective circles to meet each other in T; and join ST. So will the triangles SET, SPT, be right angled at E and P; and the parallel lines EP, *ep*, will be so near each other, that the intercepted arcs E*e*, P*p*, of the orbits, may be regarded as coinciding with the corresponding portions of their respective tangents.

Let V be the velocity of the earth in its orbit, V′ that of the planet. Then, since the arcs E*e*, P*p*, are described in the same interval of time, they will be to each other as those velocities, that is,

$$Ee : Pp :: V : V'.$$

Also, since the arcs $\mathrm{E}e$, $\mathrm{P}p$, are confounded with the portions of their respective tangents which meet at T, and the parallels EP, ep, divide TE, TP, proportionally, we have $\mathrm{ET} : \mathrm{PT} :: \mathrm{E}e : \mathrm{P}p$; and consequently,

$$\mathrm{ET} : \mathrm{PT} :: \mathrm{V} : \mathrm{V}'.$$

Put $\mathrm{SE} = a$, $\mathrm{SP} = b$, and the required angle SEP of elongation $= \mathrm{E}$. Then, since $\mathrm{SP} : \mathrm{SE} :: \sin \mathrm{E} : \sin \mathrm{EPS}$, we shall have

$$\sin \mathrm{EPS} = \cos \mathrm{EPT} = \frac{a}{b} \sin \mathrm{E}.$$

Whence, since $\sin^2 \mathrm{EPT} = \mathrm{rad}^2 - \cos^2 \mathrm{EPT}$, we have

$$\sin \mathrm{EPT} = \sqrt{\left(r^2 - \frac{a^2}{b^2} \sin^2 \mathrm{E}\right)}.$$

Farther, because $\cos \mathrm{SEP} = \sin \mathrm{PET}$, we have

$$\sin \mathrm{PET} = \sqrt{(r^2 - \sin^2 \mathrm{E})}.$$

But $\sin \mathrm{EPT} : \sin \mathrm{PET} :: \mathrm{ET} : \mathrm{PT} :: \mathrm{V} : \mathrm{V}'$.

Therefore, $\mathrm{V} : \mathrm{V}' :: \sqrt{\left(r^2 - \frac{a^2}{b^2} \sin^2 \mathrm{E}\right)} : \sqrt{(r^2 - \sin^2 \mathrm{E})}$.

Whence $\mathrm{V}^2 (r^2 - \sin^2 \mathrm{E}) = \mathrm{V}'^2 \left(r^2 - \frac{a^2}{b^2} \sin^2 \mathrm{E}\right)$;

$$\sin^2 \mathrm{E} = r^2 b^2 \frac{\mathrm{V}^2 - \mathrm{V}'^2}{\mathrm{V}^2 b^2 - \mathrm{V}'^2 a^2};$$

$$\text{and } \sin \mathrm{E} = \pm rb \sqrt{\frac{\mathrm{V}^2 - \mathrm{V}'^2}{\mathrm{V}^2 b^2 - \mathrm{V}'^2 a^2}} \ldots . (1.)$$

31. A still simpler and more convenient expression for sin E may be obtained by introducing the radii of the respective orbits for the velocities of the bodies. Thus, by the theory of central forces (*Gregory's Mechanics*, book ii. art. 282) $\mathrm{V} : \mathrm{V}' :: \sqrt{b} : \sqrt{a}$;

$$\text{we have } b : a :: r^2 - \frac{a^2}{b^2} \sin^2 \mathrm{E} : r^2 - \sin^2 \mathrm{E}.$$

Hence $(b^3 - ab^2)\, r^2 = (b^3 - a^3) \sin^2 \mathrm{E}$,

$$\text{or, } \sin^2 \mathrm{E} = \frac{(b^3 - ab^2)\, r^2}{b^3 - a^3}.$$

Supposing r and a to be each unity, we shall have

$$\sin^2 \text{E} = \frac{b^3 - b^2}{b^3 - 1} = \frac{b^2}{b^2 + b + 1},$$

$$\sin \text{E} = \pm \frac{b}{\sqrt{(b^2 + b + 1)}} \ldots\ldots (2.)$$

Also, $\cos \text{E} = \sqrt{(1 - \sin^2 \text{E})} = \sqrt{\frac{b+1}{b^2 + b + 1}} \ldots\ldots (3.)$

and $\tan \text{E} = \frac{\sin \text{E}}{\cos \text{E}} = \pm \sqrt{\frac{b^2}{b+1}} \ldots\ldots (4.)$

Here the positive sign obtains in the case of the inferior planets, the negative sign in that of the superior planets.

Example.

The mean distance of Jupiter from the sun being 5·2028 times that of the earth; required the elongation of the planet from the sun at the time of its apparent station?

The logarithmic operation for tan E indicated by equa. (4) is as follows:

From 2 log b	5·2028	1·4324742
Take log $(b + 1)$	6·2028	0·7925878
Diff. + 20		20·6398864
Its half is tan E ..	115° 35′	10·3199232

Here, the tangent being negative, it belongs to the second quadrant (chap. iv. art. 5).

Note. By like operations the mean elongations of the other planets at the time of their apparent stations may be found as below.

Mercury........	18° 12′
Venus..........	28° 51′
Mars	136° 12′
Ceres	126° 7′
Saturn	108° 47′
Uranus.........	103° 13′.

SCHOLIUM.

32. The inversion of our 4th equa. furnishes in certain cases a convenient expression by which to approximate to the mean distance of a planet from the sun.

For from $\tan E = \sqrt{\frac{b^2}{b+1}}$, we find

$$b = \tfrac{1}{2} \tan^2 E \pm \tan E \, (1 + \tfrac{1}{4} \tan^2 E)^{\frac{1}{2}}:$$

an equation which gives the radius of the planet's orbit in terms of that of the earth, from the elongation at the instant when the motion of the planet became direct after having been retrograde.

Thus, in the case of Ceres it was found by observation that E at the time of apparent station was 122° 37′ 40″; whence b was found = 3·2018. The slight difference between this elongation and the one just given, arises from this,—that the latter was actually *observed*, the planet moving in an elliptical orbit, while the former is what *would be* observed if the orbit were circular.

The following is the logarithmic operation, for finding the mean value of b in Ceres:

Log 0·25.................... $\bar{1}$·3979400
2 log tan E..................20·3873552

Sum − 20 is log $\frac{1}{4}$ tan² E ..0·60995.... $\bar{1}$·7852952
Add............1·

Log (1 + $\frac{1}{4}$ tan² E) = log ..1·60995....20·2068125

Its half......................10·1034062
Log tan E....................10·1936776

Sum − 20........log ± 1·9819...... 0·2970838
$\frac{1}{2}$ tan² E........1·2199

The sum = b = 3·2018

The negative value of 1·9819 cannot here obtain, because *b* would in that case be negative, which is impossible.

CHAPTER XI.

On the Investigation of Differential Equations for estimating the minute Variations in the Sides and Angles of Triangles.

1. BY reason of the imperfection of instruments, the unavoidable though generally slight inaccuracies of observers, the effects of parallax, refraction, the precession of the equinoxes, the varying obliquity of the ecliptic (see note, p. 119), and other causes, the sides and angles of either plane or spherical triangles, whether observed on the earth or in the heavens, can never be taken with perfect exactness. It, therefore, becomes necessary in cases where great accuracy is required, to strike out some means of estimating the extent of error which may be occasioned in certain sides and angles of triangles, by any assignable or supposable errors in the other parts. This is an interesting department of research, in which the celebrated Cotes in his treatise *De æstimatione errorum in mixtâ mathesi,* and many subsequent mathematicians have laboured with considerable ingenuity and success.

2. The inquiry before us is one (of very few in my estimation) in which the contemplation of magnitudes as augmenting and diminishing by *differences* leads to a more natural and satisfactory explication, than that in which magnitudes are regarded as varying in conse-

quence of motion. Hence I shall in this chapter employ the notation of *finite and infinitesimal differences*, instead of that of *fluxions:* although I am fully persuaded that in a great majority of mathematical inquiries the fluxional notation and metaphysics are preferable to those of differentials.

3. When variable quantities augment or diminish by portions which are finite or susceptible of mensuration, the portions which constitute the augmentation or diminution are called *differences.* If the variations, instead of being finite, are indefinitely small, they are called *differentials.* The former are aptly denoted by the capital Greek letter Δ placed before the letter which represents the variable quantity, as Δx, Δy, Δz, &c. the latter by the small or lower case Greek letter δ, as δx, δy, δz. The processes of *differentiation* and *integration* being similar to those of finding *fluxions* and *fluents*, as taught in our standard works*, need not here be explained otherwise than in connexion with the present investigation.

4. To determine the *difference*, or finite variation of the *sine* of an arc or angle, let us take from formula (U), chap. iv. the equation,

$$\sin A - \sin B = 2\sin\tfrac{1}{2}(A - B)\cos\tfrac{1}{2}(A + B).$$

Here, supposing A greater than B we may denote by ΔB, the augmentation which A must receive to become equal to B; and, in like manner, by $\Delta \sin B$, the *difference* which must be added to $\sin B$ to make it equal to $\sin A$. Thus, we shall have $A = B + \Delta B$, and $\sin A = \sin B + \Delta \sin B$. Substituting these values in the preceding equation, it becomes

$$\Delta \sin B = 2\sin\tfrac{1}{2}\Delta B\cos(B + \tfrac{1}{2}\Delta B)\ldots.(1.)$$

5. To obtain the *difference* of the *cosine*, take from the same class of formulæ the equation,

$$\cos B - \cos A = 2\sin\tfrac{1}{2}(A - B)\sin\tfrac{1}{2}(A + B);$$

---

* See the treatises on Fluxions by Maclaurin, Simpson, and Dealtry.

and, performing an operation similar to the preceding, there will result,

$$-\Delta \cos B = 2 \sin \tfrac{1}{2}\Delta B \sin (B + \tfrac{1}{2}\Delta B) \ldots . (2.)$$

6. To find the *difference* of the *tangent* of an arc or angle, assume

$$\tan A - \tan B = \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}$$

$$= \frac{\sin A \cos B - \sin B \cos A}{\cos A \cos B}$$

$$= \frac{\sin (A - B)}{\cos A \cos B};$$

and, operating as before, there will result,

$$\Delta \tan B = \frac{\sin \Delta B}{\cos B \cos (B + \Delta B)} \ldots . (3.)$$

7. Taking, in like manner,

$$\cot B - \cot A = \frac{\cos B}{\sin B} - \frac{\cos A}{\sin A} = \frac{\sin (A - B)}{\sin A \sin B};$$

we shall obtain for the *difference* of the *cotangent*

$$-\Delta \cot B = \frac{\sin \Delta B}{\sin B \sin (B + \Delta B)} \ldots (4.)$$

8. For the *difference* of the *secant*, take first,

$$\cos B - \cos A = \frac{1}{\sec B} - \frac{1}{\sec A} = \frac{\sec A - \sec B}{\sec B \sec A}$$

$$= (\sec A - \sec B) \cos B \cos A:$$

whence, form. U, chap. IV.

$$\sec A - \sec B = \frac{2 \sin \frac{1}{2}(A - B) \sin \frac{1}{2}(A + B)}{\cos A \cos B}.$$

Making in this equation substitutions analogous to the preceding, we shall have

$$\Delta \sec B = \frac{2 \sin \frac{1}{2}\Delta B \sin (B + \frac{1}{2}\Delta B)}{\cos B \cos (B + \Delta B)} \ldots . (5.)$$

9. So, again, since

$\sin A - \sin B = \frac{1}{\operatorname{cosec} A} - \frac{1}{\operatorname{cosec} B}$, we may by introducing the requisite substitutions in the value of sin A − sin B, form. U, chap. iv. obtain

$$-\Delta \operatorname{cosec} B = \frac{2\sin\frac{1}{2}\Delta B\cos(B+\frac{1}{2}\Delta B)}{\sin B\sin(B+\Delta B)} \ldots\ldots (6.)$$

10. These six equations are rigorously correct whatever the magnitude of ΔB may be. Let us trace the modifications they will undergo when the variation becomes indefinitely minute, or ΔB becomes δB. Returning to the first equation, we shall, by expanding cos $(B + \frac{1}{2}\Delta B)$ according to form. U, chap. iv. have

$$\begin{aligned}\Delta \sin B &= 2\sin\tfrac{1}{2}\Delta B\,(\cos B\cos\tfrac{1}{2}\Delta B - \sin B\sin\tfrac{1}{2}\,\tfrac{1}{2}\Delta B)\\ &= 2\sin\tfrac{1}{2}\Delta B\cos B\cos\tfrac{1}{2}\Delta B - 2\sin^2\tfrac{1}{2}\Delta B\sin B\\ &= \sin\Delta B\cos B - \sin B\,(1-\cos\Delta B).\end{aligned}$$

Now, if ΔB be indefinitely small, so as to approximate very nearly to evanescence, sin ΔB will also be indefinitely small, or practically evanescent, while cos ΔB will differ indefinitely little from radius. In that case the differential equation, will become

$$\delta \sin B = \sin\delta B\cos B - \sin B\,(1-\cos\delta B);$$

which, since $\sin\delta B = \delta B$, and $\cos\delta B = 1$, reduces to

$$\delta \sin B = \delta B\cos B \ldots\ldots (7.)$$

11. Proceeding similarly with equa. (2), we obtain

$$-\delta\cos B = \delta B\sin B,\text{ or }\delta\cos B = -\delta B\sin B \ldots\ldots (8.)$$

12. In like manner from equa. 3 and 4, we obtain

$$\delta\tan B = \frac{\delta B}{\cos^2 B} \ldots\ldots (9.)$$

$$-\delta\cot B = \frac{\delta B}{\sin^2 B},\text{ or }\delta\cot B = -\frac{\delta B}{\sin^2 B} \ldots\ldots (10.)$$

And so on, for the differentials of the secant and the cosecant.

13. Or, having found the differentials of the sine and cosine of B, others may be deduced thus:

Since versin B = 1 − cos B, we have

$$\delta \operatorname{versin} B = -\delta\cos B = \delta B\sin B \ldots\ldots (11.)$$

Also, since $\tan B = \frac{\sin B}{\cos B}$, we have

$$\delta \tan B = \frac{\cos B\ \sin B - \sin B\,\delta \cos B}{\cos^2 B}$$

$$= \frac{(\cos^2 B + \sin^2 B)\,\delta B}{\cos^2 B}.$$

But $\cos^2 B + \sin^2 B = 1$: therefore

$$\delta \tan B = \frac{\delta B}{\cos^2 B}, \text{ as before.}$$

So again, since $\cot B = \frac{1}{\tan B}$,

$$\delta \cot B = -\frac{\delta \tan B}{\tan^2 B} = \frac{-\delta B}{\tan^2 B \cos^2 B} = -\frac{\delta B}{\sin^2 B},$$

because $\tan B \cos B = \sin B$.

Farther, since $\sec B = \frac{1}{\cos B}$, we have

$$\delta \sec B = -\frac{\delta \cos B}{\cos^2 B} = \frac{\delta B \sin B}{\cos^2 B} = \frac{\delta B \tan B \cos B}{\cos^2 B}$$

$$= \frac{\delta B \tan B}{\cos B} = \delta B \tan B \sec B \ldots. (12.)$$

Lastly, since $\operatorname{cosec} B = \frac{1}{\sin B}$, we have

$$\delta \operatorname{cosec} B = -\frac{\delta \sin B}{\sin^2 B} = -\frac{\delta B \cos B}{\sin^2 B}$$

$$= -\frac{\delta B \cot B}{\sin B} = -\delta B \cot B \operatorname{cosec} B \ldots. (13.)$$

14. The *differences* and *differentials* of the principal lineo-angular quantities (chap. i. art. 4) being thus determined, we may now proceed to trace the minute variations of the six parts of triangles. In order to this, the general method consists in determining the relation of any two differentials. To determine this we must differentiate the formula which expresses the general relation of the quantities under consideration: the rule is very simple and well known to all who have studied the modern analysis.

Let the formula be $x = ayz + b$, a and b being constant quantities.

Instead of x in the first member put δx.

In the second, put δy instead of y, and you will have $\delta y \,.\, az$.

As there is a second variable quantity, put also δz for z, and you have $\delta z \,.\, ay$.

a and b, being constant quantities, furnish no variation.

Hence results the differential equation,

$$\delta x = \delta y \,.\, az + \delta z \,.\, ay;$$

$$\text{also } \frac{\delta x'}{\delta y} = az; \frac{\delta x''}{\delta z} = ay.$$

If the values of δy and δz are known, we thence know $\delta x' = \delta y \,.\, az$, and $\delta x'' = \delta z \,.\, ay$: so that the total variation of x is constituted of two parts, the one depending on δy, the other on δz.

15. In the solutions of problems relating to the variations of triangles, we have only to substitute one by one the differentials of the sine, cosine, &c. in the appropriate formula of the problem, and proceed agreeably to the above directions.

Thus, suppose that in the spherical triangle whose angles are A, B, C, and sides respectively opposite a, b, c, the sides b and c were constant. Then, from equa. (2) chap. vi. we have

$$\cos a = \cos \text{A} \sin b \sin c + \cos b \cos c.$$

This becomes by differentiation,

$$-\delta a \sin a = -\delta \text{A} \sin \text{A} \sin b \sin c.$$

The formula has only two terms, because it contains no more than two variable quantities. It results from it that

$$\frac{\delta a}{\delta \text{A}} = \frac{\sin \text{A} \sin b \sin c}{\sin a} = \frac{\sin \text{B} \sin b \sin c}{\sin b} = \frac{\sin \text{C} \sin b \sin c}{\sin c}.$$

Or, finally, that

$$\frac{\delta a}{\delta \text{A}} = \sin \text{B} \sin c = \sin \text{C} \sin b.$$

16. Upon these principles a complete series of differential equations for all possible variations of spherical

and plane triangles may be investigated. But as the inquiry would occupy more space than can be devoted to it in this work, it must be omitted. We shall here, however, lay before the reader, with a few alterations, corrections, and additions, the valuable summary of *trigonometrical differential equations* given by Delambre in his *Astronomie Théorique et Pratique.*

OBLIQUE ANGLED SPHERICAL TRIANGLES.

I. *The sides b and c constant.*

$$\frac{\delta a}{\delta A} = \sin B \sin c = \sin C \sin b$$

$$-\frac{\delta B}{\delta a} = \frac{\cot C}{\sin a} = \frac{\cos C}{\sin A \sin c}$$

$$-\frac{\delta C}{\delta a} = \frac{\cot B}{\sin a} = \frac{\cos B}{\sin A \sin b}$$

$$\frac{\delta B}{\delta C} = \frac{\tan B}{\tan C} = \tan B \cot C$$

$$-\frac{\delta B}{\delta A} = \frac{\sin B \cos C}{\sin A} = \frac{\sin b \cos C}{\sin a}$$

$$-\frac{\delta A}{\delta C} = \frac{\sin A}{\cos B \sin C} = \frac{\sin A}{\cos B \sin C}.$$

II. *The angles B and C constant.*

$$\frac{\delta A}{\delta a} = \sin b \sin C = \sin c \sin B$$

$$\frac{\delta b}{\delta A} = \frac{\cot c}{\sin A} = \frac{\cos c}{\sin C \sin a}$$

$$\frac{\delta c}{\delta A} = \frac{\cot b}{\sin A} = \frac{\cos b}{\sin a \sin B}$$

$$\frac{\delta c}{\delta b} = \frac{\tan c}{\tan b} = \tan c \cot b$$

$$\frac{\delta b}{\delta a} = \frac{\sin B \cos c}{\sin A} = \frac{\sin b \cos c}{\sin a}$$

$$\frac{\delta c}{\delta a} = \frac{\sin c \cos b}{\sin a} = \frac{\sin C \cos b}{\sin A}.$$

III. *Angle* A *and opp. side a constant.*

$$-\frac{\delta c}{\delta b} = \frac{\cos \mathrm{C}}{\cos \mathrm{B}}$$

$$\frac{\delta b}{\delta \mathrm{B}} = \frac{\tan b}{\tan \mathrm{B}}$$

$$\frac{\delta c}{\delta \mathrm{C}} = \frac{\tan c}{\tan \mathrm{C}}$$

$$-\frac{\delta \mathrm{C}}{\delta \mathrm{B}} = \frac{\cos c}{\cos b}$$

$$-\frac{\delta \mathrm{C}}{\delta b} = \frac{\sin \mathrm{C} \cot c}{\cos \mathrm{B}} = \frac{\tan \mathrm{B} \cos c}{\sin b}$$

$$-\frac{\delta c}{\delta \mathrm{B}} = \frac{\tan b \cos \mathrm{C}}{\sin \mathrm{B}} = \frac{\sin c \cot \mathrm{C}}{\cos b}.$$

IV. *Angle* A *and adj. side c constant.*

$$\frac{\delta a}{\delta b} = \cos \mathrm{C}$$

$$-\frac{\delta \mathrm{C}}{\delta a} = \frac{\tan \mathrm{C}}{\tan a} = \tan \mathrm{C} \cot a$$

$$-\frac{\delta \mathrm{C}}{\delta \mathrm{B}} = \cos a$$

$$\frac{\delta b}{\delta \mathrm{B}} = \frac{\sin a}{\sin \mathrm{C}} = \frac{\sin^2 a}{\sin \mathrm{A} \sin c} = \frac{\sin a \sin b}{\sin \mathrm{B} \sin c}$$

$$-\frac{\delta b}{\delta \mathrm{C}} = \frac{\tan a}{\sin \mathrm{C}} = \tan a \operatorname{cosec} \mathrm{C}$$

$$\frac{\delta a}{\delta \mathrm{B}} = \frac{\sin a}{\tan \mathrm{C}} = \sin a \cot \mathrm{C}.$$

Right angled Spherical Triangles.

I. *The sides b and c constant.*

This case needs no formulæ: for two constant sides with a right angle render the whole constant.

II. *Angle* B $= 90°$, *angle* C *constant.*

$$\frac{\delta \mathrm{A}}{\delta a} = \sin c = \frac{\cot \mathrm{A}}{\cot a} = \cot \mathrm{A} \tan a$$

$$\frac{\delta b}{\delta A} = \frac{2\cot b}{\sin 2A} = \frac{\cos b}{\sin a \cos A}$$

$$\frac{\delta c}{\delta A} = \frac{\cot c}{\tan A} = \frac{\cos b}{\sin B} = \frac{\cot b}{\sin A} = \frac{\cos c}{\tan a}$$

$$\frac{\delta c}{\delta b} = \cos A = \frac{\tan c}{\tan b} = \tan c \cot b$$

$$\frac{\delta b}{\delta a} = \frac{\sin 2b}{\sin 2a} = \frac{\cos c}{\sin A}$$

$$\frac{\delta c}{\delta a} = \frac{\sin 2c}{2\tan a}.$$

III. *Right angle* A *and hypoth. a constant.*

$$-\frac{\delta c}{\delta b} = \frac{\cot c}{\cot b} = \cot c \tan b$$

$$\frac{\delta b}{\delta B} = \sin c = \frac{\tan b}{\tan B}$$

$$\frac{\delta c}{\delta C} = \sin b = \frac{\tan c}{\tan C}$$

$$-\frac{\delta c}{\delta B} = \frac{\sin 2c}{\sin 2B} = \sin 2c \operatorname{cosec} 2B$$

$$-\frac{\delta c}{\delta b} = \frac{\cos c}{\cos B \sin a} = \frac{\cos c}{\cos B \sin b}$$

$$-\frac{\delta c}{\delta B} = \frac{\sin 2c}{2\cot B} = \tfrac{1}{2}\tan B \sin 2c.$$

IV. *Right angle* A *and adj. side c constant.*

$$\frac{\delta a}{\delta b} = \cos C = \sin B \cos c = \cot a \tan b$$

$$-\frac{\delta C}{\delta a} = \tan C \cot a = \cot B \operatorname{cosec} a$$

$$-\frac{\delta C}{\delta B} = \cos a = \cos b \cos c = \cot B \cot C$$

$$\frac{\delta b}{\delta B} = \frac{\sin a}{\sin C} = \frac{\sin 2b}{\sin 2B}$$

$$-\frac{\delta b}{\delta c}=\frac{\tan a}{\sin c}=\frac{2\tan b}{\sin 2c}$$

$$\frac{\delta a}{\delta B}=\frac{\sin 2a}{2\cot B}=\sin a\cot C.$$

QUADRANTAL TRIANGLES.

I. *Side b = 90°, c also constant.*

$$\frac{\delta a}{\delta A}=\frac{\cot a}{\cot A}=\cot a\tan A$$

$$\frac{\delta B}{\delta a}=\frac{2\cot B}{\sin 2a}$$

$$\frac{\delta c}{\delta a}=\frac{\cot c}{\tan a}=\cot c\cot a$$

$$\frac{\delta B}{\delta C}=\frac{\tan B}{\tan C}=\tan B\cot C$$

$$-\frac{\delta B}{\delta A}=\frac{\cos c}{\sin a}$$

$$\frac{\delta A}{\delta c}=\frac{2\tan A}{\sin 2c}.$$

II. *Two angles* B *and* C *constant.*

Here there are no formulæ, because there are necessarily three constant quantities in the complementary rectangled triangle.

III. *Side a = 90°, opp. angle* A *constant.*

$$\frac{\delta c}{\delta b}=\frac{\sin 2c}{\sin 2b}$$

$$\frac{\delta b}{\delta B}=\frac{\tan b}{\tan B}=\tan b\cot B$$

$$\frac{\delta c}{\delta C}=\frac{\tan c}{\tan C}=\tan c\cot C$$

$$-\frac{\delta c}{\delta B}=\frac{\cot c}{\cot B}=\cot c\tan B$$

$$-\frac{\delta C}{\delta b} = \frac{\sin 2B}{2\cot b}$$

$$-\frac{\delta C}{\delta B} = \frac{2\cot C}{\sin 2B}.$$

IV. *Side c* = 90°, *adj. angle* A *constant.*

$$-\frac{\delta a}{\delta b} = -\cos C = \frac{\cot a}{\tan b}$$

$$-\frac{\delta C}{\delta a} = \frac{\tan C}{\tan a} = \frac{\tan B}{\sin a}.$$

$$-\frac{\delta C}{\delta B} = \cos a = -\frac{\cot C}{\cot B}$$

$$\frac{\delta b}{\delta B} = \frac{\sin 2b}{\sin 2B}$$

$$\frac{\delta b}{\delta C} = \frac{2\cot b}{\sin 2C} = \frac{\tan a}{\sin C}$$

$$-\frac{\delta a}{\delta B} = \frac{\sin 2a}{2\tan B} = \frac{\sin a}{\tan C}.$$

PLANE TRIANGLES.

I. *Two sides b and c constant.*

$$-\frac{\delta A}{\delta B} = \frac{a}{b\cos C} = \frac{\sin A}{\sin B\cos C}$$

$$-\frac{\delta A}{\delta C} = \frac{a}{c\cos B} = \frac{\sin A}{\sin C\cos B}$$

$$\frac{\delta A}{\delta a} = \frac{1}{b\sin C} = \frac{1}{c\sin B}$$

$$\frac{\delta B}{\delta C} = \frac{\tan B}{\tan C} = \tan B\cot C$$

$$-\frac{\delta B}{\delta a} = \frac{\cot C}{a} = \frac{\cos C}{a\sin C}$$

$$-\frac{\delta C}{\delta a} = \frac{\cot B}{a} = \frac{\cos B}{a\sin B}.$$

II. *Two angles* B *and* C *constant.*

$$\frac{\delta a}{\delta c} = \frac{\sin A}{\sin C} = \frac{a}{c}$$

$$\frac{\delta a}{\delta b} = \frac{\sin A}{\sin B} = \frac{a}{b}$$

$$\frac{\delta b}{\delta c} = \frac{\sin B}{\sin C} = \frac{b}{c}.$$

A is of consequence constant.

III. *Angle* A *and opp. side a constant.*

$$-\frac{\delta B}{\delta C} = \frac{\tan C}{c}$$

$$\frac{\delta C}{\delta c} = \frac{\tan C}{c}$$

$$-\frac{\delta c}{\delta b} = \frac{\tan B}{b}$$

$$-\frac{\delta C}{\delta B} = \frac{1}{1}$$

$$-\frac{\delta c}{\delta b} = \frac{\cos C}{\cos B} = \cos C \sec B$$

IV. *Angle* A *and adj. side c constant.*

$$\frac{\delta a}{\delta b} = \cos C$$

$$-\frac{\delta C}{\delta a} = \frac{\tan C}{a}$$

$$-\frac{\delta C}{\delta B} = \frac{1}{1}$$

$$\frac{\delta b}{\delta B} = \frac{a}{\sin C} = a \operatorname{cosec} C$$

$$-\frac{\delta b}{\delta C} = \frac{a}{\sin C} = a \operatorname{cosec} C$$

$$\frac{\delta a}{\delta B} = \frac{a}{\tan C} = a \cot C.$$

17. In the preceding table the differential relations are exhibited in *equations*, both because in that form they occupy comparatively small space, and because they are in the majority of cases more commodious in application than when presented after the manner of analogies. They may, however, at once be read and regarded as analogies, by taking the numerator and denominator of the fraction in each member of the equation, as the antecedent and consequent of a ratio. Thus,

$$\frac{\delta a}{\delta b} = \cos \mathrm{C}, \text{ becomes } \delta a : \delta b :: \cos \mathrm{C} : \mathrm{rad}$$

$$-\frac{\delta \mathrm{c}}{\delta a} = \frac{\tan \mathrm{c}}{a}, \text{ becomes } \delta \mathrm{c} : -\delta a :: \tan \mathrm{c} : a,$$

and similarly of others.

18. It must be borne in mind that the equations in this table are merely *approximative*, and that the results they furnish are more accurate in proportion as the variations are small. In cases where the greatest possible correctness is required, the theorems for the *finite differences* of sines, tangents, &c. must be introduced into the appropriate formula for the individual problem, instead of those for the *differentials;* and the results employed, instead of those tabulated in the preceding pages.

We here present a few examples of the use of the formulæ above given.

Example I.

Let an object whose height is AC, be measured by taking its angle of elevation ABC, at a given horizontal distance BC from its base. It is required to ascertain what error may be committed with respect to the height AC, in consequence of any supposed error in the observed angle B?

The fourth equation, class 4 of *plane triangles*, is applicable to this example;

$$\text{viz. } \frac{\delta b}{\delta B} = \frac{a}{\sin c}, \text{ or } \frac{\delta AC}{\delta B} = \frac{BC}{\sin C} = \frac{BC}{\cos B}\text{*.}$$

But (chap. ii. 2, iv. 18), $\sin 2B = 2 \sin B \cos B$;

$$\text{whence } \frac{1}{\cos B} = \frac{2 \sin B}{\sin 2B},$$

$$\text{and, by substitution, } \frac{\delta AC}{\delta B} = \frac{2BC \sin B}{\sin 2B} = \frac{2AC}{\sin 2B}:$$

$$\text{consequently } \delta AC = \delta B\, \frac{2AC}{\sin 2B}.$$

This thrown into a proportion becomes,

As the sine of twice the observed angle,
To double the computed height;
So is the error of the observed angle,
To the corresponding error in height.

Corollary. Hence δAC, the error in the height will be a *minimum*, when $\sin 2B$ is a *maximum*, or is equal to radius. That is, the error in altitude, *cæteris paribus*, is the least when the angle of observation B is 45°.

Remark. If an error of a minute is made in taking the angle B, supposed of 45°, the corresponding error in the altitude will be its 1719th part: for in that case

$$\delta AC = \cdot0002909 \times 2AC = \frac{AC}{1718\cdot87} = \frac{AC}{1719}, \text{ nearly.}$$

Example II.

Given the inclination of the plane of a theodolite to the horizon; required the greatest error that can possibly happen in determining the magnitude of an angle.

The face of the theodolite when posited horizontally, and when inclined in any assigned angle, may be regarded as two great circles of a sphere whose centre is the centre of the face of the theodolite. Also, if the index of the theodolite is horizontal when it passes through the points 0° and 180°; when it is turned round to measure any horizontal angle upon the inclined plane, a vertical circle passing through the index in that position, would have cut the theodolite, if posited horizontally, in the true arc. Hence (sketching a right angled spherical triangle like that at p. 87) if B be the angle of inclination of the theodolite, BC or a the arc measuring the angle upon the inclined face of the instrument, then will BA or c be the arc that measures the true angle; and it is required to determine in what circumstances $a - c$ is a *maximum.*

---

* The figure, as in many other cases, is left designedly to be sketched by the student.

Now equa. 5, class 2, of *right angled spherical triangles,* is applicable to the present case, and when adapted to our figure becomes

$$\frac{\delta a}{\delta c} = \frac{\sin 2a}{\sin 2c}.$$

Hence when $\delta a - \delta c = 0$, which is the case when $a - c$ is a maximum, that is, when $\delta a = \delta c$, $\sin 2a$ must $= 2c$, and consequently $2a$ must be the supplement of $2c$, or $a + c$ must $= 90°$.

Now, (chap. vi. art. 26) $\tan c = \tan a \cos B$;

whence, $\text{rad} : \cos B :: \tan a : \tan c$

$$\left.\begin{matrix}\text{comp.}\\ \text{and div.}\end{matrix}\right\} \text{rad} + \cos B : \text{rad} - \cos B :: \tan a + \tan c : \tan a - \tan c$$

and (chap. ii. art. 18) $:: \sin (a + c) : \sin (a - c) :$ that is, $\text{rad} + \cos B : \text{rad} - \cos B :: \text{rad} : \sin (a - c)$.

From this proportion, since B is given, the measures a, c, and their difference, may be determined.

But the expression admits of farther simplification. For, transformed into an equation, it becomes

$$\frac{\sin (a - c)}{\text{rad}} = \frac{\text{rad} - \cos B}{\text{rad} + \cos B}$$

$$\text{or } \sin (a - c) = \frac{1 - \cos B}{1 + \cos B} = \tan^2 \tfrac{1}{2}B \text{ (chap. iv. R)};$$

that is, *the tangent of half the angle of inclination is a* *mean proportional between radius, and the sine of the maximum error on the theodolite.*

Suppose that the face of the theodolite, instead of being horizontal, was inclined in an angle of 5 degrees.

Then log tan $\frac{1}{2}$B .... 2° 30′.... 8·6400931
Multiplied by.......... 2

The prod. — 10, sin error 6′ 33″ 16‴.. 7·2801862

Example III.

To find when that part of the equation of time which depends on the obliquity of the ecliptic, is the greatest possible.

Here the sun's longitude will form the hypothenuse of a right angled spherical triangle, his right ascension will be the base, and the obliquity of the ecliptic is supposed constant. It is required to find when hyp. — base is a *maximum;* which is, evidently, as in the preceding example, when hyp. + base = 90°, that is, when sun's long. + sun's right ascension = 90°, from the equinoctial points.

This happens in the year 1816, about May 7 and November 8.

Example IV.

To ascertain the error that may be committed in the observation of zenith distances, by any conceivable deviation of the instrument from the vertical plane.

Draw a figure in which HO is the intersection of the horizon with the vertical plane HZO, and HZ′SO the position of the plane of the instrument, Z′ being the apparent zenith upon that plane, and S the place of the heavenly body when it has arrived at the plane of the instrument. The arc ZZ′ will measure the inclination I of the plane at H or O, and the base Z′S = Z of the right angled spherical triangle ZZ′S will be the apparent zenith distance of the body, while the hypothenuse ZS = Z + x will be the true distance.

Now from equa. 6, art. 25, chap. vi. we have

$$\cos \mathrm{ZS} = \cos \mathrm{ZZ}' \cos \mathrm{ZS}\text{: that is,}$$

$$\begin{aligned}\cos \mathrm{I} \cos \mathrm{Z} &= \cos (\mathrm{Z} + x)\\ &= \cos \mathrm{Z} \cos x - \sin \mathrm{Z} \sin x.\end{aligned}$$

Whence

$$\begin{aligned}\sin \mathrm{Z} \sin x &= \cos \mathrm{Z} (\cos x - \cos \mathrm{I})\\ &= \cos \mathrm{Z} (1 - 2\sin^2 \tfrac{1}{2}x - 1 + 2\sin^2 \tfrac{1}{2}\mathrm{I})\\ &= 2 \cos \mathrm{Z} (\sin^2 \tfrac{1}{2}\mathrm{I} - \sin^2 \tfrac{1}{2}x).\end{aligned}$$

Dividing by sin Z, there results

$$\sin x = 2 \cot \mathrm{Z} (\sin^2 \tfrac{1}{2}\mathrm{I} - \sin^2 \tfrac{1}{2}x).$$

Here since x is by hypothesis small, $\sin^2 \frac{1}{2}x$ is extremely minute, and may be rejected: hence

$$\sin x = 2 \cot \mathrm{Z} \sin^2 \tfrac{1}{2}\mathrm{I}.$$

Corol. As the zenith distance Z increases, the error x diminishes, the inclination I remaining the same.

Supposing the deviation from the vertical 5′, and the apparent zenith distance of a star 37°, the error would be $\frac{29}{100}$ of a second.

Note. If Z, Z′, be the observed zenith distances of a circumpolar star (corrected for refraction) at the superior and inferior transits of the meridian; z, z', the true zenith distances; x, x', the corrections due to the deviation of the circle from the meridian: then will

$$\begin{aligned}x &= \mathrm{Z} + x\\ x' &= \mathrm{Z}' + x';\end{aligned}$$

therefore the distance of the pole from the zenith,

$$\text{or, } \tfrac{1}{2}(z' - z) = \tfrac{1}{2}(\mathrm{Z}' + x') - \tfrac{1}{2}(\mathrm{Z} + x).$$

But in the inferior transit x' is imperceptible; therefore

$$\text{lat.} = 90° - \tfrac{1}{2}(z' - z) = 90° - \tfrac{1}{2}(\mathrm{Z}' - \mathrm{Z}) + \tfrac{1}{2}x.$$

Consequently, in the case of employing a circumpolar star, the latitude is only affected by *half* the error produced by the deviation of the instrument from the vertical plane.

Example V.

It is required to determine, at a given time and latitude, how long an interval is taken by the body of the sun to rise from the horizon.

Here the triangle which is supposed to undergo a minute variation is oblique, one of the sides being PZ, the co-latitude, or distance from the pole to the zenith, another being PS, the co-declination of the luminary at the time proposed, and the other ZS = 90° the distance from the zenith to the horizon. It is required to ascertain the variation in the hour angle P which corresponds to any assigned variation in the opposite side ZS.

The differential equation which applies to this inquiry is, obviously, the first of class 1, *oblique angled spherical triangles*, which, when accommodated to the present notation, becomes

$$\frac{\delta \mathrm{ZS}}{\delta \mathrm{P}} = \sin \mathrm{S} \sin \mathrm{PS}; \text{ or, } \delta \mathrm{P} = \frac{\delta \mathrm{ZS}}{\sin \mathrm{S} \sin \mathrm{PS}}.$$

Now, from the 2d fundamental theorem of spherical triangles, (chap. vi. p. 84), we have

$$\cos \mathrm{PZ} = \cos \mathrm{ZS} \cos \mathrm{PS} + \sin \mathrm{ZS} \sin \mathrm{PS} \cos \mathrm{S}.$$

But, in the example before us, $\cos \mathrm{ZS} = 0$, and $\sin \mathrm{ZS} = 1$. Whence $\cos \mathrm{S} = \frac{\cos \mathrm{PZ}}{\sin \mathrm{PS}}$.

$$\text{Consequently } \sin \mathrm{S} = \sqrt{\left(1 - \frac{\cos^2 \mathrm{PZ}}{\sin^2 \mathrm{PS}}\right)};$$

$$\text{and } \frac{\delta \mathrm{ZS}}{\sin \mathrm{S} \sin \mathrm{PS}} = \frac{\delta \mathrm{ZS}}{\sqrt{(\sin^2 \mathrm{PS} - \cos^2 \mathrm{PZ})}}.$$

It may be readily shown, by multiplying together the corresponding values of $\cos(\mathrm{A} + \mathrm{B})$ and $\cos(\mathrm{A} - \mathrm{B})$, (equa. c, p. 42), and substituting $1 - \cos^2$ for $\sin^2$, that

$$\sin^2 \mathrm{PS} - \cos^2 \mathrm{PZ} = \cos(\mathrm{PZ} + \mathrm{PS}) \cos(\mathrm{PZ} - \mathrm{PS}).$$

Hence it follows that

$$\delta \mathrm{P} = \frac{\delta \mathrm{ZS}}{\sqrt{[\cos(\mathrm{PZ} + \mathrm{PS}) \cos(\mathrm{PZ} - \mathrm{PS})]}}.$$

From this we have

$$\text{time sun ris.} = \frac{\text{sun's diam.} \div 15}{\sqrt{[\cos(\text{lat.} + \text{dec.}) \cos(\text{lat.} - \text{dec.})]}}.$$

Let it be proposed to ascertain the time occupied by the sun in rising from the horizon on the 25th of May, 1816, in latitude 50° 12′ N.

It appears from the Nautical Almanac, that the sun's declination on the given day is 20° 59′ N., while its apparent semidiameter is 15′ 48½″.

Hence, diam. ÷ 15 = 2·1078.

Log cos (L + D) . . 71° 11′ . . . . 9·5085850
cos (L − D) . . 29° 13′ . . . . 9·9409048

Sum ÷ 2 . . . . 19·4494898

Quotient . . . . 9·7247449
Taken from log 2·1078 . . . . 0·3238294

Rem. log 3·9727 . . . . 0·5990845

So that the time required is 3·9727 minutes, or 3ᵐ 58ˢ.

Note. By a similar theorem we may find the time which the sun's rising or setting is affected by horizontal refraction.

Example VI.

To determine the annual variation of the declination and right ascension of a fixed star, on account of the precession of the equinoxes.

Here, if A (in the spherical triangle ABC) be the pole of the ecliptic, B the pole of the equinoctial, and C the place of the star; the sides AB or c, and AC or b, must be regarded as invariable. The differential equations applicable to this question are the 1st and 2d of class I, *oblique spherical triangles*; from which the following are at once deduced.

1. Var. dec. = preces. equinox × sin obliq. eclip. × sin right ascen. from solstitial colure.

2. Var. right ascen. = $-\dfrac{\text{var. dec.} \times \text{cot ang. of posit.}}{\text{cos dec.}}$

Example VII.

To determine the variation in right ascension and declination, occasioned by any variation in the obliquity of the ecliptic.

In this example, the hypothenuse and the opposite angle of a right angled spherical triangle are assumed as constant. The 2d and 6th equations, class 3, *right angled spherical triangles*, give

1. var. dec. = var. obliq. × sin right ascen.
= var. obliq. × cot obliq. × tan dec.
2. var. right ascen. = − var. obliq. × ½ tan obliq. × sin 2 right ascen.*

CHAPTER XII.

Miscellaneous Problems.

SECTION I.

Problems with Solutions.

PROBLEM I.

REQUIRED the arc whose logarithmic tangent is 7·1644398.

First, by rule 4, p. 55.

---

* For more on this curious subject the reader may consult Cagnoli's Trigonometry, chap. xix. and xxi., and Lalande's Astronomy, vol. iii. pp. 588—604.

To .... 7·1644398
add........ 5·3144251

From .... 12·4788649
Take $\frac{2}{3}$ ar. com. log cos 3

Rem. — 10, is log 301·2067.. 2·4788646

Conseq. arc = 301″·2067 = 5′ 1″·2067.

2. By Hutton's tables.	3. By Borda's tables.
Log tan 5′ 2″ .. 7·1655821	Log tan 9′ 30″.. 7·1646031
Log tan 5′ 1″ .. 7·1641417	Log tan 9′ 20″.. 7·1599080
Difference.... 14404	Difference.... 46951
Given log tan .. 7·1644398	Given log tan .. 7·1644398
Log tan 5′ 1″ .. 7·1641417	Log tan 9′ 20″.. 7·1599080
Difference.... 2981	Difference.... 45318
arc $= 5'1'' \frac{2981}{14404} = 5'1''{\cdot}2069$	arc = decim. $9' 20'' \frac{453180}{46951}$
	$= 9' 29''{\cdot}652$
	= sexiges. $5' 1''{\cdot}2032$.*

Hence it appears that in *this* part of the tables *Hutton's* has the advantage of *Borda's* in point of accuracy. Borda, however, gives a rule to approximate more nearly to the truth; while in other parts of his tables the decimal division supplies great facilities in the use of proportional parts.

PROBLEM II.

It is required to demonstrate that if a, b, and c, represent the three sides of a plane triangle, then will

$a^2 + b^2 = c^2$, indicate a right angled triangle,
$a^2 + ab + b^2 = c^2$, one whose angle c is 120°,
$a^2 - ab + b^2 = c^2$, one whose angle c is 60°.

It appears from equa. 11, chap. iv. that

$$\cos \text{C} = \frac{a^2 + b^2 - c^2}{2ab}.$$

Substituting, then, for c^2 in this equation, its value in each of the three former, there will result, respectively,

In the 1st case, $\cos \text{C} = 0 = \cos 90^\circ$
In the 2d $\cos \text{C} = -\frac{1}{2} = \cos 120^\circ$
In the 3d $\cos \text{C} = +\frac{1}{2} = \cos 60^\circ$.

Corollary. In like manner it may be shown, that when

$$a^2 \pm \tfrac{1}{2}ab + b^2 = c^2,\ \cos \text{C} = \pm \tfrac{1}{4}, \text{ and C} = \begin{cases} 75^\circ 32' \\ 104^\circ 28' \end{cases}$$

$$a^2 \pm \tfrac{1}{3}ab + b^2 = c^2,\ \cos \text{C} = \pm \tfrac{1}{6}, \text{ and C} = \begin{cases} 80^\circ 24' \\ 99^\circ 36' \end{cases}$$

$$a^2 \pm \tfrac{1}{4}ab + b^2 = c^2,\ \cos \text{C} = \pm \tfrac{1}{8}, \text{ and C} = \begin{cases} 82^\circ 49' \\ 97^\circ 11' \end{cases}$$

$$a^2 \pm \tfrac{1}{5}ab + b^2 = c^2,\ \cos \text{C} = \pm \tfrac{1}{10}, \text{ and C} = \begin{cases} 84^\circ 16' \\ 95^\circ 44' \end{cases}$$

$a^2 \pm \frac{1}{n}ab + b^2 = c^2$, $\cos \text{C} = \pm \frac{1}{2n}$; and the two values of C supplements to each other.

PROBLEM III.

Required a commodious logarithmic method of finding the hypothenuse of a right angled plane triangle, when the base and perpendicular are given in large numbers.

B denoting the base, P the perpendicular, and H the hypothenuse;

Find N so that $2 \log \text{P} - \log \text{B} = \log \text{N}$
and make $\text{B} + \text{N} = \text{M}$.

Then, $\frac{1}{2}(\log \text{M} + \log \text{B}) = \log \text{H}$.

For, from the nature of logarithms, $\frac{\text{P}^2}{\text{B}} = \text{N}$;

also, $\frac{\text{P}^2}{\text{B}} + \text{B} = \frac{\text{P}^2 + \text{B}^2}{\text{B}} = \text{M}$; whence

$$\tfrac{1}{2}(\log \text{M} + \log \text{B}) = \tfrac{1}{2}\log \text{MB} = \tfrac{1}{2}\log\left(\frac{\text{P}^2 + \text{B}^2}{\text{B}} \times \text{B}\right)$$
$$= \log \sqrt{(\text{P}^2 + \text{B}^2)} = \log \text{H};$$ as it ought to be.

Scholium.

The following formulæ, the first three for right angled triangles, will often be of use, and may be easily demonstrated by the student.

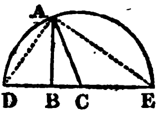
Let a and b be any two quantities, of which a is the greater. Find x, z, &c. so, that $\tan x = \sqrt{\frac{b}{a}}$, $\sin z = \sqrt{\frac{b}{a}}$, $\sec y = \frac{a}{b}$, $\tan u = \frac{b}{a}$, and $\sin t = \frac{b}{a}$: then will

$\log \sqrt{(a^2 - b^2)} = \log a + \log \sin y = \log b + \log \tan y$
$\log \sqrt{(a^2 - b^2)} = \tfrac{1}{2}[\log (a + b) + \log (a - b)]$.
$\log \sqrt{(a^2 + b^2)} = \log a + \log \sec u = \log b + \log \text{cosec}\, u$.
$\log \sqrt{(a + b)}\ \tfrac{1}{2} \log a + \log \sec x = \tfrac{1}{2} \log a + \tfrac{1}{2} \log 2 + \log \cos \tfrac{1}{2}y$.
$\log \sqrt{(a - b)} = \tfrac{1}{2} \log a + \log \cos z = \tfrac{1}{2} \log a + \tfrac{1}{2} \log 2 + \log \sin \tfrac{1}{2}y$.

$$\log (a \pm b)^{\frac{m}{n}} = \frac{m}{n}\left[\log a + \log \cos t + \log \tan \left(45^\circ \pm \tfrac{1}{2}t\right)\right].$$

PROBLEM IV.

To investigate a method of resolving quadratic equations, by means of trigonometrical tables.



1. Let $x^2 \pm px = q^2$ be the equation proposed to be resolved.

Suppose AB, and BC, the perpendicular and base of a right angled triangle ABC, to be respectively equal to q and $\frac{1}{2}p$: then

$$\text{CA} = \text{CD} = \text{CE} = \sqrt{(\text{AB}^2 + \text{BC}^2)} = \sqrt{(q^2 + \tfrac{1}{4}p^2)}$$
$$= \sqrt{(x^2 \pm px + \tfrac{1}{4}p^2)} = x \pm \tfrac{1}{2}p.$$

Consequently $x = \text{CA} \mp \frac{1}{2}p = \text{DB}$ or BE.

Now $\frac{\text{AB}}{\text{BC}} = \frac{2q}{p} = \tan \text{C} = \tan 2\text{E}$ (Euc. iii. 20.)

Also, DB : BA :: rad : tan D, or cot E, or cot $\frac{1}{2}$C
and EB : BA :: rad : tan E or tan $\frac{1}{2}$C.

$$\text{Therefore DB} = \frac{\text{BA}}{\cot \frac{1}{2}\text{C}} = \text{BA} \tan \tfrac{1}{2}\text{C} = q \tan \tfrac{1}{2}\text{C}$$

$$\text{EB} = \frac{\text{BA}}{\tan \frac{1}{2}\text{C}} = \text{BA} \cot \tfrac{1}{2}\text{C} = q \cot \tfrac{1}{2}\text{C}.$$

Which are trigonometrical values of the two roots.

2. Let $\pm px - x^2 = q^2$ be the proposed equation.

Then making $\text{AB} = q$, $\text{AC} = \frac{1}{2}p$, it may be shown, similarly to the above, that $\text{CA} \pm \text{BC} = \text{BE}$ and BD, are the two roots: whence the appropriate trigonometrical expressions may be readily deduced.

The precepts for all the cases may hence be laid down thus:—

1. If the equation be of the form $x^2 + px = q^2$:

Make $\tan \text{C} = \frac{2q}{p}$; then will the two roots be

$$x = +q \tan \tfrac{1}{2}\text{C} \;\ldots\; x = -q \cot \tfrac{1}{2}\text{C}.$$

2. For quadratics of the form $x^2 - px = q^2$.

Make, as before, $\tan \text{C} = \frac{2q}{p}$: then will

$$x = -q \tan \tfrac{1}{2}\text{C} \;\ldots\; x = +q \cot \tfrac{1}{2}\text{C}.$$

3. For quadratics of the form $x^2 + px = -(q^2)$.

Make $\sin \text{C} = \frac{2q}{p}$: then will

$$x = -q \tan \tfrac{1}{2}\text{C} \;\ldots\; x = -q \cot \tfrac{1}{2}\text{C}.$$

4. For quadratics of the form $x^2 - px = -(q^2)$.

Make $\sin \text{C} = \frac{2q}{p}$: then will

$$x = +q \tan \tfrac{1}{2}\text{C} \;\ldots\; x = +q \cot \tfrac{1}{2}\text{C}.$$

In the last two cases, if $\frac{2q}{p}$ exceed unity, sin C is imaginary, and consequently the values of x.

The logarithmic application of these formulæ is very simple, as will be manifest from the following

Example.

Find the roots of the equation $x^2 + \frac{7}{44}x = \frac{1695}{12716}$, by tables of sines and tangents.

Here $p = \frac{7}{44}$, $q^2 = \frac{1695}{12716}$, and the equation agrees with the 1st form. Also $\tan c = \frac{88}{7}\sqrt{\frac{1695}{12716}}$ in logarithms thus:

Log 1695 = 3·2291697
Arith.com.log 12716 = 5·8956495

sum + 10 = 19·1248192

half sum = 9·5624096 = log q
log 88 = 1·9444827
Arith. com. log 7 = 9·1549020

sum − 10 = log tan c = 10·6617943 = log tan 77°42′31¾″

log tan ½c = 9·9061115 = log tan 38°51′15⅞″
log q, as above = 9·5624096

sum − 10 = log x = − 1·4685211 = log ·2941176.

This value of x, viz. ·2941176, is nearly equal to $\frac{5}{17}$. To find if this be the exact root, take the arithmetical compliment of the last logarithm, viz. 0·5314379, and consider it as the logarithm of the denominator of a fraction whose numerator is unity: thus is the fraction found to be $\frac{1}{3{\cdot}4}$ exactly; which is manifestly equal to

$\frac{5}{17}$. The other root of the equation is equal to $-\frac{1695}{12716} + \frac{5}{17} = -\frac{339}{748}$.

SCHOLIUM.

Cubic equations, as well as quadratics, are readily solved by means of trigonometrical tables. The formulæ demonstrated and given by *Cagnoli* and *Borda* for this purpose, are as follow:

1. For cubics of the form $x^3 + px \pm q = 0$.

Make $\tan B = \frac{\frac{1}{3}p}{q} . 2\sqrt{\tfrac{1}{3}p} \ldots\ldots \tan A = \sqrt[3]{\tan \tfrac{1}{2}B}$.

$$\text{Then } x = \mp \cot 2A . 2\sqrt{\tfrac{1}{3}p}.$$

2. For cubics of the form $x^3 - px \pm q = 0$.

Make $\sin B = \frac{\frac{1}{3}p}{q} . 2\sqrt{\tfrac{1}{3}p} \ldots\ldots \tan A = \sqrt[3]{\tan \tfrac{1}{2}B}$.

$$\text{Then } x = \mp \operatorname{cosec} 2A . 2\sqrt{\tfrac{1}{3}p}.$$

Here, if the value of sin B should exceed unity, B would be imaginary, and the equation would fall in what is called the *irreducible case* of cubics. In that case we must make $\operatorname{cosec} 3A = \frac{\frac{1}{3}p}{q} . 2\sqrt{\tfrac{1}{3}p}$: and then the three roots would be

$$x = \pm \sin A . 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin (60^\circ - A) . 2\sqrt{\tfrac{1}{3}p}.$$
$$x = \mp \sin (60^\circ + A) . 2\sqrt{\tfrac{1}{3}p}.$$

If the value of sin B were 1, we should have $B = 90^\circ$, $\tan A = 1$; therefore $A = 45^\circ$, and $x = \mp 2\sqrt{\tfrac{1}{3}p}$. But this would not be the only root. The second solution would give $\operatorname{cosec} 3A = 1$: therefore $A = \frac{90^\circ}{3}$; and then

$$x = \pm \sin 30^\circ . 2\sqrt{\tfrac{1}{3}p} = \pm \sqrt{\tfrac{1}{3}p}.$$
$$x = \pm \sin 30^\circ . 2\sqrt{\tfrac{1}{3}p} = \pm \sqrt{\tfrac{1}{3}p}.$$
$$x = \mp \sin 90^\circ . 2\sqrt{\tfrac{1}{3}p} = +2\sqrt{\tfrac{1}{3}p}.$$

Here it is obvious that the first two roots are equal, that their sum is equal to the third with a contrary sign, and that this third is the one which is produced from the first solution.

In these solutions, the double signs in the value of x, relate to the double signs in the value of q.

PROBLEM V.

To investigate series in which any circular arc shall be expressed in terms of its sine, or of its tangent.

There are various methods of solving this problem: one of the simplest is by means of the expressions for the differentials of the sine and tangent of an arc, given in the preceding chapter.

Thus, from the expression $\partial \sin \text{B} = \partial \text{B} \cos \text{B}$, we have,

$$\partial \text{B} = \frac{\partial \sin \text{B}}{\cos \text{B}} = \frac{\partial \sin \text{B}}{\sqrt{(1 - \sin^2 \text{B})}} = \partial \sin \text{B}\,(1 - \sin^2 \text{B})^{-\frac{1}{2}}.$$

Expanding the factor $(1 - \sin^2 \text{B})$ by the binomial theorem, the whole expression becomes

$$\text{B} = \sin \text{B}\left(1 + \tfrac{1}{2}\sin^2 \text{B} + \frac{1 \cdot 3}{2 \cdot 4}\sin^4 \text{B} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\sin^6 \text{B} + \&\text{c.}\right)$$

The integration of this expression is very easy: for the integral of ∂B is B, that of $\partial \sin \text{B} \times 1$ is $\sin \text{B}$, that of $\frac{1}{2}\sin^2 \text{B}\,\partial \sin \text{B}$ is $\frac{\sin^3 \text{B}}{2 \cdot 3}$; and so on. Thus we have, at length,

$$\text{B} = \sin \text{B} + \frac{\sin^3 \text{B}}{2 \cdot 3} + \frac{3 \sin^5 \text{B}}{2 \cdot 4 \cdot 5} + \frac{3 \cdot 5 \sin^7 \text{B}}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{3 \cdot 5 \cdot 7 \sin^9 \text{B}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} + \&\text{c.}$$

Suppose B to be $= 30°$, then because $\sin 30° = \frac{1}{2}$ (ch. ii. pr. 5.) it results that

$$\text{arc } 30° = \tfrac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{3}{2 \cdot 4 \cdot 5 . 2^5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 . 2^7} +$$

$\frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9 \cdot 2^9} + \&c. = \cdot 523598775 = \frac{1}{12}$ of the circumference of a circle to the radius 1.

Again, the differential expression for the tangent is $\partial \tan B = \frac{\partial B}{\cos^2 B}$; whence

$$\partial B = \partial \tan B \cos^2 B = \frac{\partial \tan B}{\sec^2 B} = \frac{\partial \tan B}{1 + \tan^2 B} =$$

$$\partial \tan B \, (1 + \tan^2 B)^{-1}.$$

Here, expanding the factor $(1 + \tan^2 B)^{-1}$, by the binomial theorem, and integrating each term of the result, there will arise

$B = \tan B - \frac{1}{3} \tan^3 B + \frac{1}{5} \tan^5 B - \frac{1}{7} \tan^7 B + \&c.$

Thus if $B = 45^\circ$, then (ch. ii. pr. 6.) $\tan B = 1$; consequently, substituting this value of $\tan B$,

$$\text{arc } 45^\circ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}, \&c.$$

Note. This series obviously converges very slowly: *Euler* (in his *Analysis Infinitorum*, vol. i.), and *Dr. Hutton* (in his 8vo. *Tracts*, vol. i.) have transformed it into others of rapid convergence.

SCHOLIUM.

Many curious, and indeed useful, results are deducible from series of this kind, and of the opposite class, in which the sine, cosine, &c. is expressed in terms of the arc itself (ch. iv. equa. Y.) As a specimen we select the following, first discovered by *John Bernoulli*.

The expression

$$\sin x = x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{2 \cdot 3 \cdot 4 . 5 . 6 \cdot 7} + \&c.$$

when $\sin x = 0$, becomes, after dividing by x,

$$0 = 1 - \frac{1}{2 \cdot 3} x^2 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} x^4 - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} x^6 + \&c.$$

Or, making $x = \frac{1}{z}$,

$$0 = 1 - \frac{1}{2\cdot 3 z^2} + \frac{1}{2\cdot 3\cdot 4\cdot 5 z^4} - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 z^6} + \text{\&c.}$$

Multiplying by z^{2n}

$$0 = z^{2n} - \frac{1}{2\cdot 3} z^{2n-2} + \frac{1}{2\cdot 3\cdot 4\cdot 5} z^{2n-4} - \text{\&c.}$$

Now, it is shown by writers on algebra, that the sum of the roots of every equation of this form is equal to the co-efficient of the 2d term with its sign changed; that is in the present case $= \frac{1}{6}$. It is known also (chap. iv. art. 3.) that the values of x which answer to the case of $\sin x = 0$, are π, 2π, 3π, 4π, &c. π denoting the semicircumference to the radius 1. These successive values of x or of $\frac{1}{z}$ being substituted, their results

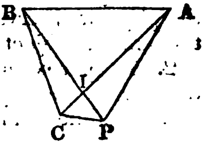
$$\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \frac{1}{4^2\pi^2} + \text{\&c.} = \frac{1}{6},$$

$$\text{or } \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{\&c.} = \frac{\pi^2}{6}.$$

PROBLEM VI.

In Geodesic operations, it is required to deduce from angles measured out of one of the stations, but near to it, the true angles at the station.

When the centre of the instrument, with which horizontal angles are taken, cannot be placed in the vertical line occupied by the axis of the signal, the angles observed must undergo a reduction, according to circumstances.



1. Let C be the centre of the station, P the place of the centre of the instrument, or the summit of the observed angle APB: it is required to find C, the measure of ACB, supposing there to be known $APB = P$, $BPC = p$, $CP = d$, $BC = L$, $AC = R$.

Since the exterior angle of a triangle is equal to the sum of the two interior opposite angles, we have, with respect to the triangle IAP, $\text{AIB} = \text{P} + \text{IAP}$; and with regard to the triangle BIC, $\text{AIB} = \text{C} + \text{CBP}$. Making these two values of AIB equal, and transposing IAP, there results

$$\text{C} = \text{P} + \text{IAP} - \text{CBP}.$$

But the triangles CAP, CBP, give

$$\sin \text{CAP} = \sin \text{IAP} = \frac{\text{CP}}{\text{AC}} \sin \text{APC} = \frac{d \sin (\text{P} + p)}{\text{R}};$$

$$\sin \text{CBP} = \frac{\text{CP}}{\text{BC}} \cdot \sin \text{BPC} = \frac{d \sin p}{\text{L}}.$$

And, as the angles CAP, CBP, are, by the hypothesis of the problem, always very small, their sines may be substituted for their arcs or measures: therefore

$$\text{C} - \text{P} = \frac{d \sin (\text{P} + p)}{\text{R}} - \frac{d \sin p}{\text{L}}.$$

Or, to have the reduction in seconds,

$$\text{C} - \text{P} = \frac{d}{\sin 1''} \left(\frac{\sin (\text{P} + p)}{\text{R}} - \frac{\sin p}{\text{L}} \right).$$

The use of this formula cannot in any case be embarrassing, provided the signs of sin p, and sin $(\text{P} + p)$ be attended to. Thus, the first term of the correction will be positive, if the angle $(\text{P} + p)$ is comprised between 0 and 180°; and it will become negative, if that angle surpass 180°. The contrary will obtain in the same circumstances with regard to the second term, which answers to the angle of direction p. The letter R denotes the distance of the object A to the right, L the distance of the object B situated to the left, and p the angle at the place of observation, between the centre of the station and the object to the left.

2. An approximate reduction to the centre may indeed be obtained by a single term; but it is not quite so correct as the form above. For, by reducing the two fractions in the second member of the last equation but one to a common denominator, the correction becomes

$$\text{C} - \text{P} = \frac{d\text{L} \sin(\text{P} + p) - d\text{R} \sin p}{\text{LR}}.$$

But the triangle ABC gives $\text{L} = \frac{\text{R} \sin \text{A}}{\sin \text{B}} = \frac{\text{R} \sin \text{A}}{\sin (\text{A} + \text{C})}$. And because P is always very nearly equal to C, the sine of A + P will differ extremely little from sin (A + C), and may therefore be substituted for it, making $\text{L} = \frac{\text{R} \sin \text{A}}{\sin (\text{A} + \text{P})}$.

Hence we manifestly have

$$\text{C} - \text{P} = \frac{d \sin \text{A} \sin (\text{P} + p) - d \sin p \sin (\text{A} + \text{P})}{\text{R} \sin \text{A}};$$

Which, by taking the expanded expressions for (P + p), and sin (A + P), and reducing to seconds, gives

$$\text{C} - \text{P} = \frac{d}{\sin 1''} \cdot \frac{\sin \text{P} \sin (\text{A} - p)}{\text{R} \sin \text{A}}.$$

3. When either of the distances R, L, becomes infinite, with respect to d, the corresponding term in the expression art. 1 of this problem, vanishes, and we have accordingly

$$\text{C} - \text{P} = -\frac{d \sin p}{\text{L} \sin 1''}, \text{ or } \text{C} - \text{P} = \frac{d \sin (\text{P} + p)}{\text{R} \sin 1''}.$$

The first of these will apply when the object A is a heavenly body, the second when B is one. When both A and B are such, then C − P = 0.

But without supposing either A or B infinite, we may have C − P = 0, or C = P in innumerable instances: that is, in all cases in which the centre P of the instrument is placed in the circumference of the circle that passes through the three points A, B, C; or when the angle BPC is equal to the angle BAC, or to BAC + 180°. Whence, though C should be inaccessible, the angle ACB may commonly be obtained by observation, without any computation. It may further be observed, that when P falls in the circumference of the circle passing through the three points A, B, C, the angles A, B, C, may be determined solely by measuring the angles APB and BPC. For, the opposite angles ABC, APC, of the quadrangle inscribed in a circle, are $= 180^\circ$. Consequently, $ABC = 180^\circ - APC$, and $BAC = 180^\circ - (ABC + ACB) = 180^\circ - (ABC + APB)$.

4. If one of the objects, viewed from a further station, be a vane or staff in the centre of a steeple, it will frequently happen that such object, when the observer comes near it, is both invisible and inaccessible. Still there are various methods of finding the exact angle at C. Suppose, for example, the signal-staff be in the centre of a circular tower, and that the angle APB was taken at P near its base. Let the tangents PT, PT′, be marked, and on them two equal and arbitrary distances P*m*, P*m*′, be measured. Bisect *mm*′ at the point *n*; and, placing there a signal-staff, measure the angle *n*PB, which, (since P*n* prolonged obviously passes through C the centre,) will be the angle p of the preceding investigation. Also, the distance P*s* added to the radius C*s* of the tower, will give $PC = d$ in the former investigation.

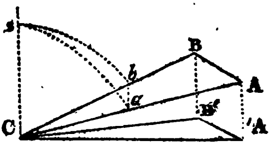
If the circumference of the tower cannot be measured, and the radius thence inferred, proceed thus: Measure the angles BPT, BPT′, then will $BPC = \frac{1}{2}(BPT + BPT') = p$; and $CPT = BPT - BPC$: Measure PT, then $PC = PT \cdot \sec CPT = d$. With the values of p and d, thus obtained, proceed as before.

5. If the base of the tower be polygonal and *regular*, as most commonly happens; assume P in the point of intersection of two of the sides prolonged, and $BPC' = \frac{1}{2}(BPT + BPT')$ as before, $PT =$ the distance from P to the middle of one of the sides whose prolongation passes through P; and hence PC is found, as above. If the figure be a regular hexagon, then the triangle P*mm*′ is equilateral, and $PC = m'm\sqrt{3}$.

PROBLEM VII.

To reduce angles measured in a plane inclined to the horizon, to the corresponding angles in the horizontal plane.

Let BCA be an angle measured in a plane inclined to the horizon, and let B′CA′ be the corresponding angle in the horizontal plane. Let d and d' be the zenith distances, or the complements of the angles of elevation ACA′, BCB′. Then from z the zenith of the observer, or of the angle C, draw the arcs za, zb, of vertical circles, measuring the zenith distances d, d', and draw the arc ab of another great circle to measure the angle C. It follows from this construction, that the angle z, of the spherical triangle zab, is equal to the horizontal angle A′C′B; and that, to find it, the three sides $za = d$, $zb = d'$, $ab = $ C, are given. Call the sum of these s; then the corresponding formula of ch. vi. pa. 90, applied to the present instance, becomes



$$\sin \tfrac{1}{2}z = \sin \tfrac{1}{2}\text{C} = \sqrt{\frac{\sin \frac{1}{2}(s - d) \sin \frac{1}{2}(s - d')}{\sin d \sin d'}}.$$

If h and h' represent the angles of altitude ACA′, BCB′, the preceding expression will become

$$\sin \tfrac{1}{2}\text{C} = \sqrt{\frac{\sin \frac{1}{2}(\text{C} + h - h') \sin \frac{1}{2}(\text{C} + h' - h)}{\cos h \cos h'}}.$$

Or, in logarithms,

$$\log \sin \tfrac{1}{2}\text{C} = \tfrac{1}{2}\,(20 + \log \sin \tfrac{1}{2}(\text{C} + h - h') + \log \sin \tfrac{1}{2}(\text{C} + h' - h) - \log \cos h - \log \cos h').$$

Cor. 1. If $h = h'$, then is $\sin \frac{1}{2}\text{C} = \frac{\sin \frac{1}{2}\text{ACB}}{\cos h}$; and

$$\log \sin \tfrac{1}{2}\text{A}'\text{CB}' = 10 + \log \sin \tfrac{1}{2}\text{ACB} - \log \cos h.$$

Cor. 2. If the angles h and h' be very small, and nearly equal; then, since the cosines of small angles vary extremely slowly, we may, without sensible error, take

$$\log \sin \tfrac{1}{2}A'CB' = 10 + \log \sin \tfrac{1}{2}ACB - \log \cos \tfrac{1}{2}(h + h').$$

Cor. 3. In this case the correction $x = A'CB' - ACB$, may be found by the expression

$$x = \sin 1'' \left(\tan \tfrac{1}{2}c \left(\tfrac{1}{4}\bigcirc - \frac{d+d'}{2}\right)^2 - \cot \tfrac{1}{2}c \left(\frac{d-d'}{2}\right)^2\right).$$

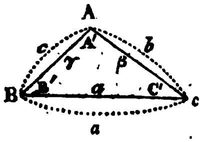
And in this formula, as well as the first given for $\sin \tfrac{1}{2}c$, d and d' may be either one or both greater or less than a quadrant; that is, the equations will obtain whether ACA' and BCB' be each an elevation or a depression.

Scholium. By means of this problem, if the altitude of a hill be found barometrically, according to one of the methods described in *Gregory's Mechanics*, vol. i. book 5, or geometrically, according to some of those described in heights and distances, (chap. v.); then, finding the angles formed at the place of observation, by any objects in the country below, and their respective angles of depression, their horizontal angles, and thence their distances, may be found, and their relative places fixed in a map of the country; taking care to have a sufficient number of angles between intersecting lines, to verify the operations.

PROBLEM VIII.

In any spherical triangle, knowing two sides and the included angle; it is required to find the angle comprehended by the chords of those two sides.

Let the angles of the spherical triangle be A, B, C, the corresponding angles included by the chords A′, B′, C′; the spherical sides opposite the former, a, b, c; the chords respectively opposite the latter, α, β, γ; then, there are given b, c, and A, to find A′.



Here, from equa. 2, chap. vi. we have

$$\cos a = \sin b \sin c \cos \text{A} + \cos b \cos c.$$

But $\cos c = \cos(\frac{1}{2}c + \frac{1}{2}c) = \cos^2 \frac{1}{2}c - \sin^2 \frac{1}{2}c$ (by art. 23, chap. iv.) $= (1 - \sin^2 \frac{1}{2}c) - \sin^2 \frac{1}{2}c = 1 - 2 \sin^2 \frac{1}{2}c$. And in like manner $\cos a = 1 - 2 \sin^2 \frac{1}{2}a$, and $\cos b = 1 - 2 \sin^2 \frac{1}{2}b$. Therefore the preceding equation becomes

$$1 - 2 \sin^2 \tfrac{1}{2}a = 4 \sin \tfrac{1}{2}b \cos \tfrac{1}{2}b \sin \tfrac{1}{2}c \cos \tfrac{1}{2}c \cos \text{A} + (1 - 2 \sin^2 \tfrac{1}{2}b)(1 - 2 \sin^2 \tfrac{1}{2}c).$$

But $\sin \frac{1}{2}a = \frac{1}{2}\alpha$, $\sin \frac{1}{2}b = \frac{1}{2}\beta$, $\sin \frac{1}{2}c = \frac{1}{2}\gamma$: which values substituted in the equation, we obtain, after a little reduction,

$$2 \times \frac{\beta^2 + \gamma^2 - \alpha^2}{4} = \beta\gamma \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \text{A} + \tfrac{1}{4}\beta\gamma^2.$$

Now, (equa. 11, chap. iv), $\cos \text{A}' = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$. Therefore, by substitution,

$$\beta\gamma \cos \text{A}' = \beta\gamma \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \text{A} + \tfrac{1}{4}\beta^2\gamma^2;$$

whence, dividing by $\beta\gamma$, there results

$$\cos \text{A}' = \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \text{A} + \tfrac{1}{2}\beta \, \tfrac{1}{2}\gamma;$$

or, lastly, by restoring the values of $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$, we have

$$\cos \text{A}' = \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c \cos \text{A} + \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c \; \ldots . (1.)$$

Cor. 1. It follows evidently from this formula, that when the spherical angle is right or obtuse, it is always *greater* than the corresponding angle of the chords.

Cor. 2. The spherical angle, if acute, is *less* than the corresponding angle of the chords, when we have $\cos \text{A}$ greater than $\frac{\sin \frac{1}{2}b \sin \frac{1}{2}c}{1 - \cos \frac{1}{2}b \cos \frac{1}{2}c}$.

PROBLEM IX.

Knowing two sides and the included angle of a rectilinear triangle, it is required to find the spherical angle of the two arcs of which those two sides are the chords.

Here β, γ, and the angle A' are given, to find A. Now, since in all cases, $\cos = \sqrt{(1 - \sin^2)}$, we have

$$\cos \tfrac{1}{2}b \cos \tfrac{1}{2}c = \sqrt{[(1 - \sin^2 \tfrac{1}{2}b)(1 - \sin^2 \tfrac{1}{2}c)]};$$

we have also, as above, $\sin \frac{1}{2}b = \frac{1}{2}\beta$, and $\sin \frac{1}{2}c = \frac{1}{2}\gamma$, Substituting these values in the equation 1 of the preceding problem, there will result, by reduction,

$$\cos A = \frac{\cos A' - \frac{1}{4}\beta\gamma}{\sqrt{(1 - \frac{1}{2}\beta) \cdot (1 + \frac{1}{2}\beta) \cdot (1 - \frac{1}{2}\gamma) \cdot (1 + \frac{1}{2}\gamma)}} \ldots (2.)$$

To compute by this formula, the values of the sides β, γ, must be reduced to the corresponding values of the chords of a circle whose radius is unity. This is easily effected by dividing the values of the sides given in feet, or toises, &c. by such a power of 10, that neither of the sides shall exceed 2, the value of the greatest chord, when radius is equal to unity.

From this investigation, and that of the preceding problem, the following corollaries may be drawn.

Cor. 1. If $c = b$, and of consequence $\gamma = \beta$, then will

$$\cos A' = \cos A \cos^2 \tfrac{1}{2}c + \sin^2 \tfrac{1}{2}c; \text{ and thence}$$

$$1 - 2 \sin^2 \tfrac{1}{2}A' = (1 - 2 \sin^2 \tfrac{1}{2}A) \cos^2 \tfrac{1}{2}c + (1 - \cos^2 \tfrac{1}{2}c):$$

from which may be deduced

$$\sin \tfrac{1}{2}A' = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}c \ldots (3.)$$

Cor. 2. Also, since $\cos \frac{1}{2}c = \sqrt{(1 - \sin^2 \frac{1}{2}c)} = \sqrt{(1 - \frac{1}{4}\gamma^2)}$, equa. 2 will, in this case, reduce to

$$\sin \tfrac{1}{2}A = \frac{\sin \frac{1}{2}A'}{\sqrt{(1 - \frac{1}{2}\gamma) \cdot (1 + \frac{1}{2}\gamma)}} \ldots (4.)$$

Cor. 3. From the equa. 3, it appears that the vertical angle of an isosceles spherical triangle, is always *greater* than the corresponding angle of the chords.

Cor. 4. If $A = 90°$, the formulæ 1, 2, give

$$\cos A' = \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c = \tfrac{1}{4}\beta\gamma \ldots (5.)$$

These five formulæ are strict and rigorous, whatever be the magnitude of the triangle. But if the triangles be small, the arcs may be put instead of the sines in equa. 5, then

Cor. 5. As $\cos A' = \sin (90° - A') =$ in this case, $90° - A'$; the small excess of the spherical right angle over the corresponding rectilinear angle, will, supposing the arcs b, c, taken in seconds, be given in seconds by the following expression:

$$90° - A' = \frac{\frac{1}{4}bc}{R''} = \frac{bc}{4R''} \ldots (6.)$$

The error in this formula will not amount to a second, when $b + c$ is less than $10°$, or than 700 miles measured on the earth's surface.

Cor. 6. If the hypothenuse does not exceed $1\frac{1}{2}°$, we may substitute $a \sin \mathrm{C}$ instead of c, and $a \cos \mathrm{C}$ instead of b; this will give $bc = a^2 \sin \mathrm{C} \cos \mathrm{C} = \frac{1}{2}a^2 \sin 2\,(90° - \mathrm{B}) = \frac{1}{2}a^2 \sin 2\mathrm{B}$: whence

$$(90° - \mathrm{A}') = \frac{a^2 \sin 2\mathrm{C}}{8\mathrm{R}''} = \frac{a^2 \sin 2\mathrm{B}}{8\mathrm{R}''} \ldots\ldots (7.)$$

If $a = 1\frac{1}{2}°$, and $\mathrm{B} = \mathrm{C} = 45°$ nearly; then will $90° - \mathrm{A}' = 17''{\cdot}7$.

Cor. 7. Retaining the same hypothesis of $\mathrm{A} = 90°$, and $a =$ or $< 1\frac{1}{2}°$, we have

$$\mathrm{B} - \mathrm{B}' = \frac{b^2 \cot \mathrm{B}}{8\mathrm{R}''} = \frac{bc}{8\mathrm{R}''} \ldots\ldots (8.)$$

$$\text{Also } \mathrm{C} - \mathrm{C}' = \frac{bc}{8\mathrm{R}''} \ldots\ldots (9.)$$

Cor. 8. Comparing formulæ 8, 9, with 6, we have $\mathrm{B} - \mathrm{B}' = \mathrm{C} - \mathrm{C}' = \frac{1}{2}\,(90° - \mathrm{A}')$. Whence it appears that the sum of the two excesses of the oblique spherical angles, over the corresponding angles of the chords, in a small right angled triangle, is equal to the excess of the right angle over the corresponding angle of the chords. So that either of the formulæ 6, 7, 8, 9, will suffice to determine the difference of each of the three angles of a small right angled spherical triangle, from the corresponding angles of the chords. And hence *this* method may be applied to the measuring an arc of the meridian by means of a series of triangles.

PROBLEM X.

In a spherical triangle ABC, right angled in A, knowing the hypothenuse BC (*less than* 4°) and the angle B, it is required to find the error e committed through finding by plane trigonometry, the opposite side AC.

Referring still to the diagram of prob. 8, where we now suppose the spherical angle A to be right, we have (equa. 5, chap. vi) $\sin b = \sin a \sin B$. But it has been proved (chap. iv. equa. Y) that the sine of any arc A is equal to the sum of the following series:

$$\sin A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$$

$$\text{or, } \sin A = A - \frac{A^3}{6} + \frac{A^5}{120} - \frac{A^7}{5040} + \&c.$$

And, in the present inquiry, all the terms after the second may be neglected, because the 5th power of an arc of 4° divided by 120, gives a quotient not exceeding 0″·01. Consequently, we may assume $\sin b = b - \frac{1}{6}b^3$, $\sin a = a - \frac{1}{6}a^3$; and thus the preceding equation will become

$$b - \tfrac{1}{6}b^3 = \sin B \left(a - \tfrac{1}{6}a^3\right)$$

$$\text{or, } b = a \sin B - \tfrac{1}{6}\left(a^3 \sin B - b^3\right).$$

Now, if the triangle were considered as rectilinear, we should have $b = a \sin B$; a theorem which manifestly gives the side b or AC too great by $\frac{1}{6}(a^3 \sin B - b^3)$. But, neglecting quantities of the fifth order, for the reason already assigned, the last equation but one gives $b^3 = a^3 \sin^3 B$. Therefore, by substitution, $e = -\frac{1}{6}a^3 \sin B\,(1 - \sin^2 B)$: or, to have this error in seconds, take R'' = the radius expressed in seconds, so shall

$$e = -a \sin B \,.\, \frac{a^2 \cos^2 B}{6R''R''}.$$

Cor. 1. If $a = 4°$, and $B = 35° \, 16'$, in which case the value of $\sin B \cos^2 B$ is a maximum, we shall find $e = -4\frac{1}{2}''$.

Cor. 2. If, with the same data, the correction be applied, to find the side c adjacent to the given angle, we should have

$$e' = a \cos B \, \frac{a^2 \sin^2 B}{3R''R''}.$$

So that this error exists in a contrary sense to the other; the one being subtractive, the other additive.

Cor. 3. The data being the same, if we have to find the angle C, the error to be corrected will be

$$e'' = a^2 \frac{\sin 2\text{B}}{4\text{R}''}.$$

As to the excess of the arc over its chord, it is easy to find it correctly from the expressions in prob. 8: but for arcs that are very small, compared with the radius, a near approximation to that excess will be found in the same measures as the radius of the earth, by taking $\frac{1}{24}$ of the quotient of the cube of the length of the arc divided by the square of the radius.

PROBLEM XI.

It is required to investigate a theorem, by means of which, spherical triangles, whose sides are small compared with the radius, may be solved by the rules for plane trigonometry, without considering the chords of the respective arcs or sides.

The sum of the angles of a spherical triangle exceeds 180°, by a quantity E called (chap. vi. p. 102) *the spherical excess*, which is a correct measure of the area of the triangle.

The sum of the angles of a spherical triangle ABC, exceeds 180° by $2r^2 \cdot \frac{1}{2}\text{AC} \cdot \frac{1}{2}\text{AB} \sin \text{A} = \frac{1}{2}r^2\text{AB} \cdot \text{AC} \sin \text{A}$ = surface of the triangle; r being the radius of the sphere. Or, when $r = 1$, we have

$$\begin{aligned} \text{A} + \text{B} + \text{C} &= 180° + \tfrac{1}{2}\text{AB} \cdot \text{AC} \sin \text{A} \\ &= 180° + \tfrac{1}{2}\text{AB} \cdot \text{CB} \sin \text{B} \\ &= 180° + \tfrac{1}{2}\text{AC} \cdot \text{BC} \sin \text{C} \\ &= 180° + \tfrac{1}{6}\text{AB} \cdot \text{AC} \sin \text{A} + \tfrac{1}{6}\text{AB} \cdot \text{CB} \sin \text{B} \\ &\quad + \tfrac{1}{6}\text{AC} \cdot \text{BC} \sin \text{C}. \end{aligned}$$

Distributing this excess equally, by taking its third part from each angle, we shall have

$$\begin{aligned} \text{A}' &= \text{A} - \tfrac{1}{6}\text{AB} \cdot \text{AC} \sin \text{A}, \\ \text{B}' &= \text{B} - \tfrac{1}{6}\text{AB} \cdot \text{CB} \sin \text{B}, \\ \text{C}' &= \text{C} - \tfrac{1}{6}\text{AC} \cdot \text{BC} \sin \text{C}: \end{aligned}$$

and the following steps may readily be traced by those who have carefully studied chapters iv. and vi. of this work:—

$$\text{AC} = \frac{\text{BC} \sin \text{B}'}{\sin \text{A}'} = \frac{\text{BC} \sin (\text{B} - \tfrac{1}{6}\text{AB} \, . \, \text{CB} \sin \text{B})}{\sin (\text{A} - \tfrac{1}{6}\text{AB} \, . \, \text{AC} \sin \text{A})}$$

$$= \frac{\text{BC} \sin (\text{B} - \tfrac{1}{3}\text{E})}{\sin (\text{A} - \tfrac{1}{3}\text{E})}$$

$$= \frac{\text{BC} \sin \text{B} \cos \tfrac{1}{3}\text{E} - \text{BC} \cos \text{B} \sin \tfrac{1}{3}\text{E}}{\sin \text{A} \cos \tfrac{1}{3}\text{E} - \cos \text{A} \sin \tfrac{1}{3}\text{E}}$$

$$= \left[\frac{\text{BC} \sin \text{B}}{\sin \text{A}} - \frac{\text{BC} \cos \text{B} \sin \tfrac{1}{3}\text{E}}{\sin \text{A} \cos \tfrac{1}{3}\text{E}}\right] \div (1 - \cot \text{A} \tan \tfrac{1}{3}\text{E})$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} \cdot \frac{1 - \cot \text{B} \tan \tfrac{1}{3}\text{E}}{1 - \cot \text{A} \tan \tfrac{1}{3}\text{E}}$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} (1 + \tan \tfrac{1}{3}\text{E} \cot \text{A} - \tan \tfrac{1}{3}\text{E} \cot \text{B})$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} (1 + \tfrac{1}{6}\text{AB} \, . \, \text{AC} \sin \text{A} \cot \text{A} - \tfrac{1}{6}\text{AB} \, . \, \text{BC} \sin \text{B} \cot \text{B})$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} (1 + \tfrac{1}{6}\text{AB} \, . \, \text{AC} \cos \text{A} - \tfrac{1}{6}\text{AB} \, . \, \text{BC} \cos \text{B})$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} (1 + \tfrac{1}{6}\text{AB} \, . \, \text{AD} - \tfrac{1}{6}\text{AB} \, . \, \text{BD})$$

[AD and BD being segments of the base AB by a perpendicular CD from C]

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} [1 + \tfrac{1}{6}\text{AB} (\text{AD} - \text{BD})]$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} [1 + \tfrac{1}{6}\text{AD}^2 - \tfrac{1}{6}\text{BD}^2]$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} (1 + \tfrac{1}{6}\text{AC}^2 - \tfrac{1}{6}\text{BC}^2)$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} + \frac{\text{BC} \, . \, \text{AC}^2 \sin \text{B}}{6 \sin \text{A}} - \frac{\text{BC}^3 \sin \text{B}}{6 \sin \text{A}}$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} + \frac{\text{AC}^3 \sin \text{A}}{6 \sin \text{A}} - \frac{\text{BC}^3 \sin \text{B}}{6 \sin \text{A}}$$

$$= \frac{\text{BC} \sin \text{B}}{\sin \text{A}} + \tfrac{1}{6}\text{AC}^3 - \frac{\text{BC}^3 \sin \text{B}}{6 \sin \text{A}}.$$

Hence, $AC - \frac{1}{6}AC^3 = (BC - \frac{1}{6}BC^3)\,\frac{\sin B}{\sin A}$.

But (chap. iv. equa. Y) $\sin AC = AC - \frac{1}{6}AC^3$, nearly, and $\sin BC = BC - \frac{1}{6}BC^3$, nearly; the error, as shown prob. 10, being imperceptible in small arcs. Consequently,

$$\sin AC = \frac{\sin BC \sin B}{\sin A}.$$

And thus we have deduced the truth of the following theorem, viz.

A spherical triangle being proposed, of which the sides are very small, compared with the radius of the sphere; if from each of its angles one third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, whose sides are equal in length to those of the proposed spherical triangle.[*]

SCHOLIUM.

We have already given (in sect. 6, chap. vi.) expressions for finding the spherical excess. A few additional rules may with propriety be presented here.

1. The spherical excess E, may be found in seconds, by the expression $E = \frac{R''s}{r}$; where s is the surface of the triangle (whose sides are a, b, c, and angles A, B, C) $= \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}a^2 . \frac{\sin B \sin C}{\sin (B+C)}$, r is the radius of the earth, in the same measures as a, b, and c, and $R'' = 206264''{\cdot}8$, the seconds in an arc equal in length to the radius.

If this formula be applied logarithmically, then $\log R'' = \log \frac{1}{\text{arc } 1''} = 5{\cdot}3144251$.

---

[*] This curious theorem was first announced by M. Legendre, in the Memoirs of the Paris Academy, for 1787. The investigation here given is by M. Delambre.

2. From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant log 9·3267737, then the remainder is the logarithm of the excess above 180°, in seconds nearly.*

3. Since $S = \frac{1}{2}bc \sin A$, we shall manifestly have $E = \frac{R''}{2r^2} bc \sin A$. Hence, if from the vertical angle B we demit the perpendicular BD upon the base AC, dividing it into two segments α, β, we shall have $b = \alpha + \beta$, and thence $E = \frac{R''}{2r^2} c\,(\alpha + \beta) \sin A = \frac{R''}{2r^2} \alpha c \,.\, \sin A + \frac{R''}{2r^2} \beta c \,.\, \sin A$. But the two right angled triangles ABD, CBD, being nearly rectilinear, give $\alpha = a \cos C$, and $\beta = c \cos A$; whence we have

$$E = \frac{R''}{2r^2} ac \sin A \cos C + \frac{R''}{2r^2} c^2 \sin A \cos A.$$

In like manner, the triangle ABC, which itself is so small as to differ but little from a plane triangle, gives $c \sin A = a \sin C$. Also, $\sin A \cos A = \frac{1}{2} \sin 2A$, and $\sin C \cos C = \frac{1}{2} \sin 2C$ (equa. I, chap. iv). Therefore, finally,

$$E = \frac{R''}{4r^2} a^2 \sin 2C + \frac{R''}{4r^2} c^2 \sin 2A.$$

From this theorem a table may be formed, from which the spherical excess may be found; entering the table with each of the sides above the base and its adjacent angle, as arguments.

4. If the base b and height h, of the triangle are given, then we have evidently $E = \frac{1}{2}bh \frac{R''}{r^2}$. Hence results the following simple logarithmic rule: add the logarithm of the base of the triangle, taken in feet, to the logarithm of the perpendicular, taken in the same measure; deduct from the sum the logarithm 9·6278037; the re-

---

* This is Mr. Dalby's rule given by General Roy in the Philosophical Transactions, for 1790, p. 171.

mainder will be the common logarithm of the spherical excess in seconds and decimals.

5. Lastly, when the three sides of the triangle are given in feet; add to the logarithm of half their sum, the logs. of the three differences of those sides and that half sum, divide the total of these 4 logs. by 2, and from the quotient subtract the log 9·3267737; the remainder will be the logarithm of the spherical excess in seconds, &c. as before.

One or other of these rules will apply to all cases in which the spherical excess will be required.*

PROBLEM XII.

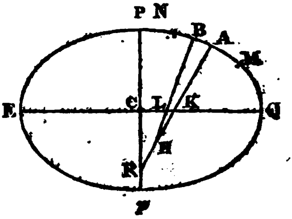
To determine the ratio of the earth's axes from the measures of convenient portions of a meridian in any two given latitudes; the earth being supposed a spheroid generated by the rotation of an ellipse upon its minor axis.

The most accurate way of solving this problem, will be to compare, not merely single degrees measured on different parts of the meridian, but large portions of 5, 6, or 7 degrees, the most extensive that have been correctly measured; according to the method proposed by Professor *Playfair* (*Edinburgh Transactions*, vol. v.), which is as below.

Let the ellipse PE*p*Q represent a terrestrial meridian passing through the poles P, *p*, and cutting the equator in E, Q. Let C be the centre of the earth, CQ the radius of the equator $= a$, and PC, half the polar axis $= b$.

---

* The intelligent student who wishes to go more minutely into the subject of geodesic operations, especially in reference to the determination of the figure and magnitude of the earth, may consult the chapter on that subject in the 3d. vol. of Dr. Hutton's Course, Colonel Mudge's "Account of the Trigonometrical Survey of England and Wales," M. Puissant's works, entitled "Geodésie" and "Traité de Topographie, d'Arpentage, &c." Mechain and Delambre, "Base du Système Métrique Décimal," and chap. XXXV. of Delambre's quarto "Astronomie."



Let AB be any small arc of the meridian, having its centre of curvature in H; join HA, HB, intersecting CQ in K and L. Let φ be the measure of the latitude of A, or the measure of the angle QKA, expressed in decimals of the radius 1; not in degrees and minutes. Then, the excess of the angle QLB above QKA, that is, the angle LHK or BHA will be $= \partial\varphi$. Also, if the elliptic arc QA $= z$, then will AB $= \partial z = \partial\varphi \times$ AH.

Now it is shown by the writers on conic sections, that the radius of curvature at A, that is,

$$\text{AH} = \frac{a^2b^2}{(a^2 - a^2\sin^2\varphi + b^2\sin^2\varphi)^{\frac{3}{2}}} = a^2b^2\,(a^2 - a^2\sin^2\varphi + b^2\sin^2\varphi)^{-\frac{3}{2}}.$$

If, therefore, c be the compression at the poles, or $= a - b$, we shall have $b^2 = a^2 - 2ac + c^2$; or, rejecting the powers of c higher than the first, because c is very small in comparison to a (that is, about the 300th part), it will be $b^2 = a^2 - 2ac$, whence, by substitution,

$$\text{AH} = a^3\,(a - 2c)\,(a^2 - a^2\sin^2\varphi + a^2\sin^2\varphi - 2ac\sin^2\varphi)^{-\frac{3}{2}} = a^3\,(a - 2c)\,(a^2 - 2ac\sin^2\varphi)^{-\frac{3}{2}}.$$

But, $(a^2 - 2ac\sin^2\varphi)^{-\frac{3}{2}} = a^{-3}\left(1 - \frac{2c}{a}\sin^2\varphi\right)^{-\frac{3}{2}}$

$$= a^{-3}\left(1 + \frac{3c}{a}\sin^2\varphi\right) \text{ nearly};$$

rejecting, as before, the terms that involve c^2, c^3, &c.

Hence AH $= (a - 2c)\left(1 + \frac{3c}{a}\sin^2\varphi\right)$

$$= a - 2c + 3c\sin^2\varphi.$$

Hence, also,

$$\partial z = \partial\varphi \times \text{AH} = \partial\varphi\,(a - 2c + 3c\sin^2\varphi)$$
$$= (a - 2c)\,\partial\varphi + 3c\sin^2\varphi\,\partial\varphi,$$

which latter, because $\sin^2 \varphi = \frac{1}{2}(1 - \cos 2\varphi)$, becomes

$$\partial z = (a - 2c)\,\partial\varphi + \tfrac{3}{2}c\,\partial\varphi - \tfrac{3}{2}c\cos 2\varphi\,\partial\varphi.$$

Consequently, by integration,

$$z = (a - \tfrac{1}{2}c)\,\varphi - \tfrac{3}{4}c\sin 2\varphi = a\varphi - \tfrac{1}{2}c\,(\varphi + \tfrac{3}{2}\sin 2\varphi);$$

an expression which needs no correction.

Let MN be any arc of the elliptic meridian, φ' the latitude of M, one of its extremities, and φ'' that of N, its other extremity; we have

$$\begin{aligned} \text{QM} &= a\varphi' - \tfrac{1}{2}c\,(\varphi' + \tfrac{3}{2}\sin 2\varphi') \\ \text{QN} &= a\varphi'' - \tfrac{1}{2}c\,(\varphi'' + \tfrac{3}{2}\sin 2\varphi'') \end{aligned}$$

and their difference QN — QM, that is,

$$\text{MN} = a\,(\varphi'' - \varphi') - \tfrac{1}{2}c\,[(\varphi'' - \varphi') + \tfrac{3}{2}(\sin 2\varphi'' - \sin 2\varphi')].$$

If, therefore, MN be an arc of several degrees of the meridian, the length of which is known by actual measurement, and the latitude of its two extremities M and N also known, the above formula for MN becomes an equation in which a and c are the only unknown quantities. In like manner by the measurement of another arc of the meridian, another equation will be obtained, in which a and c are the only unknown quantities. Therefore, by comparing these two equations, the values of a and c, that is, of the radius of the equator, and its excess above half the polar axis, may be determined.

Thus, if l be the length of an arc measured, m the coefficient of a, and n of c, computed by the last formula; also, if l' be the length of any other arc, m' the coefficient of a, and n' of c, computed in the same manner; then we shall have

$$ma - nc = l, \text{ and } m'a - n'c = l'.$$

Whence $a = \dfrac{n'l - nl'}{mn' - m'n}$; $c = \dfrac{m'l - ml'}{mn' - m'n}$;

$$\text{and } \frac{c}{a} = \frac{m'l - ml'}{n'l - nl'}.$$

It is, also, evident that $c = \dfrac{ma - l}{n}$.

Note. From these theorems, Professor Playfair, by comparing an arch of 3° 7′ 1″ measured in Peru, with an arc of 8° 20′ 2½″ measured from Dunkirk to Perpignan, found $\frac{c}{a} = \frac{1}{300}$ nearly. And from the same theorems, I, by comparing the arc in Peru with the arc from Dunnose to Clifton, in Yorkshire, amounting to 2° 50′ 23⅓″, obtained $\frac{1}{332\cdot68}$ for the resulting compression. (See my Collection of Dissertations and Letters relating to the Trigonometrical Survey of England and Wales, pa. 47.) There is great reason to conclude that the true compression lies between these limits.

SECTION II.

Problems without Solutions.

1. Demonstrate the truth of the following analogy, viz. As the sine of half the difference of two arcs which together make 60° or 90° respectively, is to the difference of their sines; so is 1 to $\surd 3$ or $\surd 2$, respectively.

2. Demonstrate that 4 times the rectangle of the sines of two arcs, is equal to the difference of the squares of the chords of the sum and difference of those arcs.

3. Demonstrate that of any arc A,

$$\sin A = \frac{1}{\surd(1 + \cot^2 A)} = \frac{\tan A}{\surd(1 + \tan^2 A)}.$$

4. Also, that

$$\cos A = \frac{1}{\surd(1 + \tan^2 A)} = \frac{\cot A}{\surd(1 + \cot^2 A)}.$$

5. And that

$$\tan A = \sqrt{\left(\frac{1}{\cos^2 A} - 1\right)} = \frac{2 \tan \frac{1}{2}A}{1 - \tan^2 \frac{1}{2}A}$$

$$= \frac{2}{\cot \frac{1}{2}A - \tan \frac{1}{2}A} = \frac{2 \cot \frac{1}{2}A}{\cot^2 \frac{1}{2}A - 1}.$$

6. Demonstrate that

$$\tan^2 A - \tan^2 B = \frac{\sin (A - B) \sin (A + B)}{\cos^2 A \cos^2 B}.$$

7. Also, that

$$\cot^2 B - \cot^2 A = \frac{\sin (A - B) \sin (A + B)}{\sin^2 A \sin^2 B}.$$

8. Demonstrate that

$$\log \sin (A + B) = \log \sin A + \log \sin B$$
$$+ \log (\cot A + \cot B) - 20.$$

9. Demonstrate that

$$\sec 2A = \frac{\sec^2 A}{2 - \sec^2 A};$$

$$\sec 3A = \frac{\sec^3 A}{4 - 3 \sec^2 A};$$

$$\sec 4A = \frac{\sec^4 A}{8 - 8 \sec^2 A + \sec^4 A};$$

$$\sec 5A = \frac{\sec^5 A}{16 - 20 \sec^2 A + 5 \sec^4 A}.$$

10. Demonstrate that

$$\text{arc to tan } \tfrac{1}{2} + \text{arc to tan } \tfrac{1}{3} = \text{arc } 45^\circ$$
$$\text{arc to tan } \tfrac{1}{7} + 2 \text{ arc to tan } \tfrac{1}{3} = \text{arc } 45^\circ.$$

11. Demonstrate that the tangent of the sum of any number of arcs will be represented by

$$\frac{A - C + E - G \ \&c.}{1 - B + D - F \ \&c.}$$

the sum of all the tangents of the separate arcs being denoted by A, the sum of all their rectangles by B, the sum of all their solids by C, &c.

12. Find the arc whose secant and co-tangent shall be equal.

Ans. Arc whose sine is $\frac{1}{2} \sqrt{5} - \frac{1}{2}$, or 38° 10′.

13. Find the arc whose sine added to its cosine shall be equal to a; and show the limits of possibility.

Ans. a^2 must never exceed 2.

14. What arc is that whose tangent and cotangent shall together be equal to four times radius?

Ans. Arc of 75° or of 15°.

15. Given the two segments of the base of a plane triangle made by a perpendicular from the vertical angle, equal to 9 and 1 respectively, and the vertical angle = 55° 46′ 16″: required the other angles and sides, by means of cor. to prop. 18. ch. ii.

Ans. Angles, 40° 24′ 34½″ and 82° 49′ 9″:
Sides, 8 and 12.

16. In a plane triangle are given, the perimeter = 1502, and the sines of the angles respectively as 232, 174, and 345: required the sides and angles.

Ans. Sides, 464, 348, 690:
Angles, 27° 3′, 37° 20′, 115° 37′.

17. If one angle of a plane triangle be 129° 34′, and the two sides about that angle in the ratio of 4 to 7; what are the other angles?

Ans. 32° 41′ 7″ and 17° 44′ 53″.

18. At what distance on the horizontal plane passing through the foot of a steeple of 50 yards in height must an observer stand, that the said steeple may give an apparent elevation of 30 degrees?

Ans. 86⅗ yards.

19. A statue of 12 feet high stands at the top of a column of 48 feet high. At what distance from the base of the column, on the same horizontal plane, will the statue appear under the greatest possible angle, and what will that angle be?

Ans. Distance 53·6656 feet, angle 6° 22′ 46″.

20. In a right angled triangle ABC, right angled at B, there are known a right line AE bisecting the angle A and falling on the opposite side BC, = 4, and a right line CD bisecting the angle C and terminating in AB, = 2: required the three sides of the triangle?

Ans. AB = 3·9193, BC = 1·6683, AC = 4·2596.

21. The respective distances from a given point P to the three nearest angles of a square garden ABCD, are PA = 70 yards, PB = 40, and PC = 60. Required the length of a side of that garden.

Ans. 31.4885 yards.

22. A gentleman travelling towards York, discovered, by means of a telescope, the top of the tower of the cathedral, when just in the horizon; and on travelling 20 miles nearer, he found the top of the tower elevated one degree. What is its height, that of the observer's eye being 5 feet 6 inches?

Ans. 260¾ feet.

23. There are four trees, A, B, C, D, standing in a straight hedge row; their distances are AB = 60 yards, BC = 20, CD = 40. Where must I stand to observe them, so that the three intervals may appear equal?

Ans. At 12 yards from B towards C, erect to the line ABCD a perpendicular MP of 24 yards; then will the angles APB, BPC, CPD, be each = 45°.

24. Having at a certain unknown distance taken the angle of elevation of a steeple, I advanced 60 yards nearer (upon level ground), and then observed the elevation to be the complement of the former: advancing still 20 yards nearer, the elevation was found to be double the first. Hence the height of the steeple is required?

Ans. 74·16198 yards.

25. The excess of the three angles of a triangle, measured on the earth's surface, above two right angles, is 1 second: what is its area, taking the earth's diameter at 7957¾ miles?

Ans. 76·75299 square miles.

26. The three sides of a triangle, measured on the earth's surface, and reduced to the level of the sea, are 17, 18, and 10 miles. What is the spherical excess?

27. The angles subtended by two distant objects at a third object is 66° 30′ 39″; one of those objects appeared under an elevation of 25′ 47″; the other under a depression of 1″. Required the reduced horizontal angle?

Ans. 66° 30′ 37″.

28. In two right angled spherical triangles ABC, ADE, having one angle A common, let there be given the two

perpendiculars BC, DE, and the sum or difference of the hypothenuses AC, AE; then it will be,

$$\tan \tfrac{1}{2}(DE + BC) : \tan \tfrac{1}{2}(DE - BC) :: \tan \tfrac{1}{2}(AE + AC) : \tan \tfrac{1}{2}(AE - AC).$$

Required a demonstration.

29. If, besides the two perpendiculars, there be given the sum or difference of the bases, AB, AD; then it will be,

$$\sin(DE + BC) : \sin(DE - BC) :: \tan \tfrac{1}{2}(AD + AB) \tan \tfrac{1}{2}(AD - AB).$$

Required a demonstration.

30. Given the hypothenuse AC, and the sum AB + BC, or difference AB − BC of the legs of a right angled spherical triangle: then it will be,

$$2 \cos AC = \cos(AB + BC) + \cos(AB - BC),$$

whence the legs become known. Required a demonstration.

31. If, in a plane triangle ABC there be given, the sides AC, BC, and the line CD drawn to bisect the vertical angle and terminating in the base: then we have

$$AB = (AC + BC)\sqrt{\frac{AC \cdot BC - CD^2}{AC \cdot BC}};$$

$$\text{and } \cos ACD = \cos BCD = \frac{(AC + BC)CD}{2AC \cdot BC}.$$

Required a demonstration.

32. If the triangle ABC be spherical, and the same parts be given; then

$$\cos ACD = \cos BCD = \frac{\sin(AC + BC)\tan CD}{2 \sin AC \sin BC}.$$

Required a demonstration.

33. Given the north latitude of three places, viz. Moscow 55° 30′, Vienna 48° 12′, Gibraltar 35° 30′, all lying directly in the same arc of a great circle. The difference of longitude between Vienna (situated between the other two) and Moscow, easterly, is equal to that between Vienna and Gibraltar, westerly. It is required

to find the true bearing and distance of each place from the other, as well as their difference of longitude.

Ans. Difference of longitude between Moscow and Vienna 14° 13′ 27″, between Moscow and Gibraltar 28° 26′ 54″.

Vienna and Gibraltar bear from Moscow. S. 56° 4′ W.
Moscow from Vienna................ N. 44° 50′ E.
Gibraltar from Vienna .... S. 44° 50′ W.
Vienna and Moscow from Gibraltar .... N. 35° 16′ E.

Vienna is distant from Gibraltar 16° 29′ = 1146 English miles; Vienna from Moscow, 11° 23′ = 791 miles; Gibraltar from Moscow, 27° 52′ = 1987 miles.

34. Two sides of a plane triangle are 80 and 50, and the line bisecting the verticle angle and terminating in the base is 52. Required the angles of the triangle, without previously finding the base.

Ans. 37° 38′ 16″, 64° 39′ 22″, and 77° 42′ 22″.

35. Two sides of a plane triangle are 12 and 9, and the right line drawn from the vertical angle to the middle of the third side is 8. Required the angles of the triangle.

Ans. 39° 45′ 10″, 58° 29′ 55″, and 81° 44′ 55″.

36. When all the three sides of any plane triangle are given, an angle A may be found by the following theorem:

$$\text{versin } A = \frac{2\left(\frac{1}{2}\text{ perim} - AB\right)\left(\frac{1}{2}\text{ perim} - AC\right)}{AB \cdot AC}.$$

Required a demonstration.

37. In any spherical triangle ABC,

$$\text{versin } A = \frac{\text{versin } CB - \text{versin }(CA - BA)}{\sin AC \sin AB}.$$

38. Also,

$$\sin^2 \tfrac{1}{2}A = \frac{\frac{1}{2}\left[\text{versin } CB - \text{versin }(CA - BA)\right]}{\sin AC \sin AB}.$$

39. And

$$\tan^2 \tfrac{1}{2}A = \frac{\text{versin } CB - \text{versin }(CA - BA)}{\text{versin }(CA + BA) - \text{versin } CB}.$$

Required the demonstration.

40. On a certain horizontal dial the sum of the angular distances of the hour lines of 3 and 4 in the afternoon from the 12 o'clock hour line, is $82^\circ 48'$. For what latitude was the dial made?

Ans. The latitude of $41^\circ 48'$.

41. On another horizontal dial the distance between the hour lines of 4 and 5 o'clock in the afternoon was 20 degrees. For what latitude was this dial made?

Ans. The latitude of $35^\circ 53'$.

42. In what latitude is the angle included between the hour lines of 12 and 1, on a horizontal dial, double of that included between the same hour lines on a vertical south dial?

Ans. Latitude $63^\circ 45' 20''$.

43. Suppose a horizontal dial made for the latitude of $51^\circ 32'$, to be fixed horizontally in the latitude of 54°; what will be the greatest error, and at what time of the day?

Ans. The greatest error will be $4^m 27^s$ of time, and it will occur at $6^h 15^m 27^s$, true time from noon.

44. On what day of the year, in the latitude of London, will the length of the afternoon exceed that of the morning by the greatest difference possible; conceiving the day to begin at sun-rising and to end at sun-setting?

Ans. When the sun's longitude is $29^\circ 32'$ from the first point of Aries, that is, on the 19th day of April; the afternoon of that day exceeding the morning by $1^m 4\frac{1}{2}^s$.

45. In latitude 53° N. stands a tower, the shadow of whose summit on June 10th, 1816, described on the horizontal plane a hyperbolic curve whose transverse axis was 150 yards. What was the height of that tower?

Ans. 43·688 yards.

46. Required the latitude of the place and the declination of the sun when the length of the day is to that of the night as 3 to 2, and the sun's mid-day altitude to his midnight depression as 2 to 1?

Ans. Lat. $61^\circ 57'$ N. Declination $9^\circ 21'$ N.

47. In the north latitude of 51° 30′ I observed the sun to rise on a certain point of the compass; and eleven hours after the sun had arisen, I perceived my shadow to be projected towards the same point. What was the sun's declination?

Ans. Declination 9° 54′ 41″ N.

48. To what height must a person be raised above the city of London, on June 21st, at midnight, so that he may just see the sun's upper limb?

Ans. 155·4283 miles, the radius of the earth being supposed 3980.

49. In the latitudes of 30° and 50° north, on the same meridian, on the 21st of June in the morning, it is required to determine the exact instant when the sun's altitude will be equal, if observed at both places; also the latitude at that time?

Ans. Time $3^h\ 55^m\ 16^s$ from noon; altitude 37° 37′ 17″.

50. At what time of the year is the night (exclusive of twilight) longer at York (N. lat. 54°) than it is either at London (N. lat. 51° 32′) or at Edinburgh (N. lat. 56° 7′)?

Ans. Whilst the sun's declination south is between 14° 14′ 46″ and 14° 40′ 34″; that is, on February 10th and November 1st.

51. In what north latitude may an erect south declining dial be fixed, to have the declination of the plane, the distance of the substyle from the meridian, and the height of the style, all equal?

Ans. The latitude of 38° 2′; and the plane's declination will be 38° 13′.

52. In the spring quarter last year, day broke at 3 o'clock, and the sun's altitude that morning when due east was 32° 42′. Where and when did this happen?

Ans. Lat. 38° 39′ 20″ N.; Dec. 19° 43′ 18″ N. answering to May the 19th.

53. At a certain place I observed the sun to rise at 10 minutes past 4 o'clock, and his altitude at noon to be

58° 40′. What were the latitude and day of observation?

Ans. Lat. 51° 30′ N.; Dec. 20° 10′ N., answering to May 21st and July 22d.

54. On the longest day last year it was observed at a certain place, that the sun's altitude when due east was 14° 46′ more than it was at 6 o'clock the same morning; what were those altitudes, and what was the latitude of the place?

Ans. Altitudes 32° 8′ 5″ and 17° 22′ 5″,
Latitude 48° 30′ 49″ N.

55. What is the northern latitude, time of the year, and time of the day, in 1816, when the sun's altitude, his azimuth from the east, the arc from noon, and the co-latitude of the place are equal to each other?

Ans. Lat. 51° 28′ 53″, time of year Apr 17, or Aug. 26,
time of day 9ʰ 25ᵐ 56ˢ A.M.

56. In what north latitude is the shortest day equal to $\frac{8}{15}$ of the longest at London?

Ans. Latitude 43° 28′.

57. In what latitude and time of the year does the sun rise at half past 5 o'clock, and appear due east at 10?

Ans. Lat. 21° 13′, Dec. 18° 35′, both of the same kind.

58. Where is the sun's altitude at 6 o'clock, on the longest day, equal to the co-latitude?

Ans. N. latitude 68° 17′ 12″.

59. To find the declination of that star whose change in azimuth is a *maximum* or *minimum* in a given time, reckoned from the time that it transits a given almucantar in a given latitude. Suppose the latitude of London, the time one hour, and the almucantar 15° above the horizon.

Ans. Star's declination 20° 25′ south.

60. What arc of a circle is equal to its tangent?

Ans. Arcs of 257° 27′ 12″, 442° 37′ 28″, 624° 45′ 38″, 805° 56′ 1″, 986° 40′ 36″, 1167° 11′ 28″, 1347° 33′ 55″, 1527° 51′ 9″, 1708° 5′ 44″, or 1888° 16′ 12″, answer the conditions of the question; and these arcs fall in the 2d, 4th, 6th, 8th, 10th, 12th, &c. quadrants, running continually round the same circle.

61. What arc is that whose sine shall be equal to the sine of triple the same arc?

62. What arc is that whose sine is half the sine of its triple arc?

63. What arc is that whose sine is the $\frac{1}{n}$th part of the sine of its triple arc?

Ans. The arc whose sine is $\frac{1}{2}\sqrt{(3-n)}$.

64. What arc is that whose sine is equal to the sine of its quadruple arc?

65. What arc is that whose sine is equal to the sine of its quintuple arc?

66. What arc is that whose sine is equal to n times its cosine?

Ans. The arc whose sine is $\sqrt{\frac{n^2}{n^2+1}}$.

67. What arc is that whose secant is equal to n times its tangent?

Ans. The arc whose tangent is $\sqrt{\frac{1}{n^2-1}}$.

68. In a right angled triangle the right lines drawn from the acute angles to the middle points of the opposite sides are equal to a and b respectively: required the acute angles.

Ans. The acute angles are those whose tangents are $\sqrt{\frac{4a^2-b^2}{4b^2-a^2}}$ and $\sqrt{\frac{4b^2-a^2}{4a^2-b^2}}$.

Thus if $a = 4$, $b = 3$, then the tangents are $\sqrt{\frac{11}{4}}$ and $\sqrt{\frac{4}{11}}$, and the angles are 58° 54′ 32″ and 31° 5′ 28″.

69. Demonstrate the truth of Dr. Maskelyne's four

rules for determining the log. sines and tangents of small arcs, given at pa. 55 of this volume.

70. Supposing the latitude of London to be 51° 30′ N., the latitude and longitude of Moscow 55° 45′ N. and 38° E., and the latitude and longitude of Constantinople 41° 30′ N. and 29° 15′ E. It is required to determine the latitude and longitude of a place which shall be equidistant from the former three.

Ans. Lat. 51° 17′ N., long. 19° 13′ E.

71. Three stars A, B, and C, were all observed to be in the arc of a great circle; the distance of A and B was found to be 10°, of B and C 20°; the difference of the azimuths of A and C was found to be 90°; and the middlemost B was the least distance possible from the zenith. Required the altitudes of the three stars?

Ans. Alt. of A, 72° 18′ 14″, of B, 75° 19′ 32″, and of C, 65° 22′ 33″.

72. In what north latitude will the sun appear due east, on the longest day, at the mid-time between sun-rise and noon?

Ans. In N. lat. 64° 35′ 48.″

73. Let ABC be a plane triangle of which $\frac{BC}{AC} = m$; it is required to demonstrate, that the value of the angle A is expressed in seconds by the first or second of the following series, according as the perpendicular from B on AC falls within or out of the triangle: viz.

$$A = \frac{m \sin C}{\sin 1''} + \frac{m^2 \sin 2C}{\sin 2''} + \frac{m^3 \sin 3C}{\sin 3''} + \frac{m^4 \sin 4C}{\sin 4''} + \&c.$$

$$A = \frac{m \sin C}{\sin 1''} - \frac{m^2 \sin 2C}{\sin 2''} + \frac{m^3 \sin 3C}{\sin 3''} - \frac{m^4 \sin 4C}{\sin 4''} + \&c.$$

74. Demonstrate also that

$$A = \frac{m \cos B}{\sin 1''} + \frac{m^2 \sin 2B}{\sin 2''} - \frac{m^3 \cos 3B}{\sin 3''} - \frac{m^4 \sin 4B}{\sin 4''} + \&c.$$

$$A = \frac{m \cos B}{\sin 1''} - \frac{m^2 \sin 2B}{\sin 2''} - \frac{m^3 \cos 3B}{\sin 3''} + \frac{m^4 \sin 4B}{\sin 4''} + \&c.$$

75. If in a plane triangle $\tan \text{B} = n \tan \text{A}$, then demonstrate that

$$(\text{A} - \text{B}) = \frac{1-n}{1+n} \cdot \frac{\sin 2\text{A}}{\sin 1''} - \left(\frac{1-n}{1+n}\right)^2 \cdot \frac{\sin 4\text{A}}{\sin 2''}$$
$$+ \left(\frac{1-n}{1+n}\right)^3 \cdot \frac{\sin 6\text{A}}{\sin 3''} - \&\text{c}.$$

and that if $n = \cos \text{C}$, then

$$(\text{A} - \text{B}) = \frac{\tan^2 \frac{1}{2}\text{C} \sin 2\text{A}}{\sin 1''} - \frac{\tan^4 \frac{1}{2}\text{C} \sin 4\text{A}}{\sin 2''}$$
$$+ \frac{\tan^6 \frac{1}{2}\text{C} \sin 6\text{A}}{\sin 3''} - \&\text{c}.$$

76. Let E be the spherical excess in seconds of a spherical triangle ABC, then it is to be demonstrated that

$$\tfrac{1}{2}\text{E} = \frac{\tan \frac{1}{2}b \tan \frac{1}{2}c \sin \text{A}}{\sin 1''} - \frac{\tan^2 \frac{1}{2}b \tan^2 \frac{1}{2}c \sin 2\text{A}}{\sin 2''}$$
$$+ \frac{\tan^3 \frac{1}{2}b \tan^3 \frac{1}{2}c \sin 3\text{A}}{\sin 3''} - \&\text{c}.$$

77. Demonstrate also, that

$$\cot \tfrac{1}{2}\text{E} = \cot \text{A} + \frac{\cot \frac{1}{2}b \cot \frac{1}{2}c}{\sin \text{A}}:$$

and find E, when $\text{A} = b = c = 179°$.

Ans. $\text{E} = 357° 59' 58''$.

78. Given the latitude of the place, and the position of two hour circles, with respect to the meridian; to determine the declination of that star whose change in altitude shall be the greatest possible in passing over the interval between those hour circles.

Ans. Let h' be the greatest, and h the least hour angle from the meridian, L the latitude, and D the declination; then

$$\tan \text{D} = \tan \text{L} \frac{\sin \frac{1}{2}(h' - h)}{\sin \frac{1}{2}(h + h')}.$$

79. If a person could approach so near to the moon as to see one third of her convex surface, what angle

at his eye would the diameter of her apparent disc subtend?

Ans. Twice the angle whose natural cosecant is thrice radius, that is, twice 19° 28′ 16″ or 38° 56′ 32″.

80. In a plane triangle ABC are given

$$\cot A \cot B = \tfrac{1}{2},\ \cot B \cot C = \frac{1}{6};$$

to find the angles A, B, and C.

Ans. 45°, 63° 26′ 6″, and 71° 33′ 54″.

81. In a certain north latitude it was observed that on the morning of the longest day the body of the sun occupied precisely 8 minutes of time in rising out of the horizon. What was the latitude?

Ans. 61° 29′ 54″ disregarding refraction and parallax: or, 62° 28′ 54″, taking the horizontal refraction at 33′, and the parallax at 9″.

THE END.

C. Baldwin, Printer,
New Bridge-street, London.

Publications by the Author of this Work.

1. LETTERS TO A FRIEND, on the Evidences, Doctrines, and Duties of the CHRISTIAN RELIGION. Third Edition, in two vols. 8vo.; price 14*s.* boards.

2. LESSONS ASTRONOMICAL AND PHILOSOPHICAL, for the Amusement and Instruction of British Youth, 12mo. Fourth Edition, price 5*s.* boards.

3. A TREATISE ON ASTRONOMY, in which the Elements of the Science are deduced in a natural Order, from the Appearances of the Heavens to an Observer on the Earth; demonstrated on Mathematical Principles, and explained by an Application to the various Phenomena. One large volume 8vo. price 9*s.* boards.

4. A TREATISE OF MECHANICS, Theoretical, Practical, and Descriptive; containing, besides a comprehensive View of the Theory, illustrative Descriptions of at least 200 curious and useful Machines. Two volumes 8vo. with a volume of plates. 2*l.* 2*s.* boards, third edition in the press.

5. A TRANSLATION OF M. HAÜY'S ELEMENTARY TREATISE ON NATURAL PHILOSOPHY, with many Notes and Illustrations. 2 vols. 8vo.; price 14*s.* boards.

⁂ The three latter Works constitute together a complete System of Natural Philosophy.

6. In conjunction with Dr. *Hutton*, the third volume of A COURSE OF MATHEMATICS for the Use of the Gentlemen Cadets in the Royal Military Academy, Woolwich. Price 12*s.*

7. In conjunction with *John Mason Good*, Esq. F.R.S and Mr. *Newton Bosworth*, the PANTALOGIA, a Universal Dictionary of Arts, Sciences, and Words: published in monthly parts, at 6*s.* each: and completed in 12 vols. royal 8vo.

BOOKS OF EDUCATION

Published by

BALDWIN, CRADOCK, AND JOY,

47, PATERNOSTER-ROW.

By Dr. KELLY, Master of the Finsbury Square Academy.

1. **THE ELEMENTS OF BOOK-KEEPING**; comprising a System of Merchants' Accounts, founded on Real Business, and adapted to Modern Practice. With an Appendix, on Exchanges, Banking, and other Commercial Subjects, in 8vo. price 7*s.* bound.

2. RULED BOOKS, on the best Writing Paper, adapted to Dr. Kelly's Elements of Book-keeping, viz. 1. A Waste-book and Journal, and Ledger; adapted to the SECOND SET; price 5*s.* 2. A Waste-book and Cash-book, Invoice-book and Sales-book, Bill-book, Journal, and Ledger; adapted to the THIRD SET, price 12*s.* 6*d.*

3. A PRACTICAL INTRODUCTION TO SPHERICS AND NAUTICAL ASTRONOMY; being an Attempt to simplify those useful Sciences. With an Appendix, on Time, Time-keepers, Transit Instruments, &c. Fourth Edition, in royal 8vo.; 12*s.* boards.

By Mr. GUY, Late Professor of Geography, &c. Royal Military College, Great Marlow.

1. **THE POCKET CYCLOPÆDIA**; or Miscellany of Useful Knowledge, from the latest and best Authorities; designed for senior Scholars in Schools, and for young Persons in general; containing much useful Information on various Subjects necessary to be known by all Persons, and yet not to be found in Books of General Use in Schools. Seventh Edition, on fine paper, handsomely printed; with the addition of a copious List of Authorities, to which the Student is referred for more extended Information; price 7*s.* 6*d.* boards, or 8*s.* neatly bound.

2. A NEW BRITISH PRIMER, for Children of an early Age; intended to precede THE NEW BRITISH SPELLING BOOK, and other similar Reading Books. The Fourth Edition, with many new cuts, and much improved; price 6*d*. sewed.

3. THE NEW BRITISH SPELLING BOOK, or an Introduction to Spelling and Reading, in Seven Parts, containing a great Variety of easy Lessons, exactly adapted to the Capacities of Youth, and arranged in a new, easy, and pleasing Order; the Tables of Words are divided and accented according to the purest Modes of Pronunciation. The Fourteenth Edition, on fine paper, and ornamented with numerous cuts; price 1*s*, 6*d*, bound.

4. THE NEW BRITISH READER, or SEQUEL to the NEW BRITISH SPELLING BOOK; containing a great Variety of easy Lessons, selected from the most approved Authors; exhibiting a very easy Gradation, and adapted to the junior Classes of Ladies' and Gentlemen's Schools. The difficult Words are accented and explained at the Head of each Lesson. The Second Edition, enlarged; price 4*s*. bound.

5. THE SCHOOL CIPHERING BOOK, for Beginners; containing a complete Set of Sums in the first four Rules of Arithmetic, printed in large Figures, the Copy-book Size, having all the sums set, and all the Lines ruled. The Second Edition, on an excellent writing paper; 3*s*. 6*d*. half-bound.

Also, a KEY to the above, price 6*d*.

6. A CHART OF GENERAL HISTORY, Ancient and Modern. A Chart of this kind will greatly facilitate the student's progress, and give him clearer ideas of the rise, duration, and fall of each kingdom and empire, than the perusal of many volumes. It is, in short, to *History*, what Maps are to Geography. It is a mode of representation which gives a kind of locality to events; and conveys not only distinct ideas of distant events in any one country, but the relative occurrences of different nations. Third Edition, on a sheet of Columbier drawing paper, price 7*s*. coloured; on canvas and rollers, 10*s*. 6*d*.; and varnished, 14*s*.

7. THE SCHOOL GEOGRAPHY; on a new, easy, and highly improved plan; comprising not only a complete general Description, but much Topographical Information; expressly adapted to every Class of Learners, both in Ladies' and Gentlemen's Schools. Royal 18mo. with seven correct Maps. A New Edition; price 3*s*. bound in red.

The maps, on a handsome royal drawing paper, neatly coloured, are done up separately; price 2*s*.

8. A GEOGRAPHICAL KEY to the Problems and Questions in *Guy's Geography*. Price 1*s*. 6*d*. sewed.

By Mr. GUY, Junior.

1. THE ENGLISH SCHOOL GRAMMAR; in which practical Illustration is, in every step, blended with Theory, by Rules, Examples, and Exercises, adapted throughout to the Use of Schools and Private Teachers. The Third Edition, enlarged, very considerably improved, and handsomely printed; price 1*s.* 6*d.* bound in red.

2. THE NEW LATIN PRIMER: or a Companion to Latin Grammars, in three parts. 1. Contains regular Nouns, Adjectives, Pronouns to be declined, and Verbs to be conjugated according to the Examples of those parts of Speech in the Accidence. 2. Contains the Exceptions to the three special Rules for the Genders; and the Irregulars in Declension and Conjugation, so arranged as to be well understood and retained. 3. Exhibits such irregular Declensions and Conjugations at length, as are not usually found in School Grammars, 18mo. price 2*s.* bound.

By Mr. GOODACRE, of Standard-Hill Academy.

1. ARITHMETIC; adapted to different Classes of Learners, but more particularly to the Use of large Schools. In Three Parts. Arranged in a new manner, and enlivened with numerous original Examples on interesting Subjects. With an Appendix, containing Five Classes of Recapitulatory Exercises. The Fifth Edition, price 4*s.* bound.

2. A KEY TO GOODACRE'S ARITHMETIC, containing Answers to every Question in that Work; with the Solutions, nearly at full length, of more than One Thousand of the most important Examples, interspersed with explanatory Notes and Observations. The Third Edition, very much enlarged; price 5*s.* 6*d.* neatly bound.

3. A TREATISE ON BOOK-KEEPING, adapted to the Use of Schools; containing two Sets of Books by Single Entry, one by Double Entry, and an Outline Set to be filled up by either Method: To which is added, a familiar Dissertation on the various Bills and Notes used in Commerce as Substitutes for Cash. 12mo.; price 4*s.* bound.

4. TABLES OF ARITHMETIC. Price, per dozen, 1*s.*

By Mr. HOLLAND, of Bolton.

1. ESSAYS ON ANCIENT HISTORY; particularly the Jewish, Assyrian, Persian, Grecian, and Roman: with Exami-

nations, for the Use of young Persons. A New Edition, with extensive alterations and Additions. 12mo. 6*s*. bound.

2. A SYSTEM OF ANCIENT AND MODERN GEOGRAPHY, with a Series of Geographical Examinations. The Fifth Edition, corrected and much improved; price 4*s*.

*** A peculiar feature of this work consists in a variety of Questions for Examination, both in Ancient and Modern Geography.

3. DEFINITIONS, MAXIMS, PROVERBS, AND PRECEPTS on Grammar, Arithmetic, Natural Philosophy, Geography, &c. proposed for COPIES in PENMANSHIP, and for Exercises in Grammar and Composition. The Second Edition; price 1*s*. sewed.

"The wise lay up knowledge." SOLOMON.

4. QUESTIONS ON GENERAL HISTORY, Ancient and Modern; comprising Questions on the Jewish, Egyptian, and Grecian History—the Roman History—on the Decline and Fall of the Roman Empire—the Dark Ages, and on Modern History, and the Revival of Learning. In 12mo.; price 1*s*. sewed.

5. A SERIES OF QUESTIONS ON THE HOLY SCRIPTURES, with Reference to the Bible for Answers. Price 3*s*. half-bound.

6. EXERCISES FOR THE MEMORY AND UNDERSTANDING, with a series of Examinations. Fourth Edition, consisting of Fables and Narratives, Selections from Natural and Civil History, and Moral and Religious Extracts, in Prose and Verse. 12mo. bound, 5*s*. 6*d*.

By the Rev. THOMAS PEACOCK, *of Denton, near Darlington.*

1. THE TUTOR'S ASSISTANT MODERNIZED, or a regular System of Practical Arithmetic; comprising all the modern Improvements in that Art which are necessary for the Man of Business and the practical Scholar. Third Edition, much improved; price 4*s*. neatly bound.

2. THE PRACTICAL MEASURER; containing the Uses of Logarithms, Gunter's Scale, the Carpenter's Rule, and the Sliding Rules: the best and most approved Modes of drawing Geometrical Figures; the Doctrine of Plane Trigonometry, and its Application to Heights and Distances; the Mensuration of Superficies, Solids, and Artificers' Works; and the Methods of Surveying, Planning, and Dividing Land. The Second Edition, illustrated with Plates, 12mo.; price 4*s*. 6*d*. bound.

By Mr. JOHN GREIG.

1. THE YOUNG LADIES' NEW GUIDE TO ARITHMETIC; containing the common Rules and Questions on Domestic Affairs, with the Applications of each Rule, the Method of making Bills of Parcels, Receipts, Notes, &c. The Eighth Edition, enlarged, and stereotyped. 2*s.* bound.

2. GREIG'S WORLD DISPLAYED: or, the Characteristic Features of Nature and Art exhibited, on a new Plan; intended for Youth in general, as an Outline of the most striking Parts of Human Knowledge, and as a Remembrancer to those of riper Years. In a 12mo. volume, with an elegant Frontispiece, price 8*s.* 6*d.* bound; or on large paper, for Families, price 12*s.* 6*d.* boards.

3. AN INTRODUCTION TO THE USE OF THE GLOBES; containing Definitions and Problems in Geometry, the Stereographic Projection of the Sphere, the Rise and Progress of Geography and Astronomy. 12mo. 3*s.* bound.

By the Rev. J. JOYCE.

1. SCIENTIFIC DIALOGUES; intended for the Instruction and Entertainment of Young People: in which the First Principles of Natural and Experimental Philosophy are fully explained. In 6 vols. 15s. half-bound. Comprising,

Vol. I. Mechanics,
II. Astronomy,
III. Hydrostatics,
IV. Pneumatics,
Vol. V. Optics and Magnetism,
VI. Electricity and Galvanism.

2. A COMPANION TO THE SCIENTIFIC DIALOGUES; or, The Tutor's Assistant and Pupil's Manual in Natural and Experimental Philosophy: containing a complete Set of Questions, and other Exercises, for the Examination of Pupils, in the said Volumes of Scientific Dialogues, and forming a seventh Volume of that Work. To which is added, A Compendium of the principal Facts under each Department of Science. Half-bound, 2*s.* 6*d.*

3. DIALOGUES ON CHEMISTRY, intended for the Instruction and Entertainment of Young People; in which the First Principles of that Science are fully explained. To which are added, Questions and other Exercises for the Examination of Pupils. New Edition, in 2 vols. price, half-bound, 7*s.*

4. LETTERS on Natural and Experimental Philosophy, Chemistry, Anatomy, Physiology, and other Branches of Science pertaining to the material World. Illustrated with 19 plates, 12mo. boards, 10*s.* 6*d.*